MISCELLANEOUS
TRACTS

ON
Some curious, and very intereſting SUBJECTS
IN
MECHANICS, PHYSICAL-ASTRONOMY, and SPECULATIVE MATHEMATICS;

WHEREIN,
The Preceſſion of the EQUINOX, the Nutation of the EARTH'S AXIS, and the Motion of the MOON in her ORBIT, are determined.

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BRITISH MUSEUM

TO THE
RIGHT HONOURABLE
THE EARL OF
MACCLESFIELD, &c.
PRESIDENT of the ROYAL SOCIETY.

MY LORD,

WHATEVER Luſtre, to the public Eye, Works of Learning may derive from the Patronage of the Great, it is to your Lordſhip's perſonal Acquirements, and extenſive Knowledge in the Mathematical Sciences, that my Ambition of deſiring leave to prefix your Name to this Performance, is to be imputed: And indeed, My Lord, an Author's natural Partiality permits me not to hope, or wiſh, that any thing theſe ſheets contain, will meet with a more general Approbation, than what is due to the Propriety of their being inſcribed to THE EARL OF MACCLESFIELD.

DEDICATION.

Were your Character, My Lord, leſs conſpicuous and diſtinguiſh'd, the Obligations I have to your Lordſhip's Goodneſs, would, alone, be Motives ſufficient to make me gladly embrace this Opportunity of publicly expreſſing my warmeſt Gratitude, and of teſtifying the perfect Eſteem, and profoundeſt Deference, with which I am,

My Lord,

Your Lordſhip's

moſt Obliged,

and ever Obedient

Humble Servant

Thomas Simpſon.

PREFACE.

THE Tracts, *or* Papers *compoſing the* Work *here offered to the Publick, were drawn up at ſeveral, diſtant times, and upon different occaſions; either, with a view to clear up, or ſettle ſome difficult or controverted point in* Aſtronomy, *to ſhew the conformity of* Theory *with* Obſervations; *or to extend and facilitate the* analytic-method *of computation, by ſome improvements and applications, that have not at all, or but ſlightly, been touched upon, at leaſt by any Engliſh Author.*

The firſt *of theſe* Papers, *which is one of the moſt conſiderable in the whole work, is concerned in determining the* Preceſſion of the Equinox, *and the various inequalities thereof, with the different motions of nutation of the Earth's Axis, ariſing from the attraction of the ſun and moon; wherein the late important diſcovery of* Dr. Bradley, *relating to an apparent motion of the* Fix'd Stars, *unknown to former* Aſtronomers, *is ſhewn to be intirely conſiſtent with the Theory of Gravitation.*—*This piece was drawn up about five years ago, in conſequence of another on the ſame ſubject, by* M. Silvabelle *(a French Gentleman) then delivered to me, for my opinion, ſince printed in the* Philoſophical Tranſactions.—*Tho' I have particular reaſons for mentioning this circumſtance, I would not be thought to inſinuate here, that my opinion had any weight with Thoſe to whom the publication of that paper was owing: I have, indeed, no reaſon to believe it.*—*Tho' the author thereof had gone through one* part *of the ſubject with ſucceſs and perſpicuity, and though his concluſions were found perfectly conformable to* Dr. Bradley's *obſervations, He nevertheleſs appeared (and ſtill appears) to me: to have greatly failed in a very material, and indeed the only very difficult* part, *that is, in the determination of the* momentary alteration *of the poſition of the earth's axis, cauſed by the forces of the* Sun *and* Moon; *of which forces, the quantities, but not the effects, are truly inveſtigated.*

The Second Paper, *contains the inveſtigation of an eaſy, and very exact method, or rule, for finding the place of a* Planet *in its* Orbit, *from a correction of* Dr. Ward's *circular hypotheſis; by means of certain* Equations *applied to the motion about the upper focus of the ellipſis. From whence that table of* Dr. Halley's, *entitled,* Tabula pro expediendo calculo Æquationis centri Lunæ, *may be very readily conſtructed.*

ſtructed.—By this method, the reſult, even in the orbit of Mercury, *may be found within a* ſecond *of the truth, without repeating the operation.*

The Third, *ſhews the manner of transferring the motion of a* Comet *from a* parabolic, *to an* elliptic Orbit; *being of great uſe, when the* obſerved Places *of a (new)* Comet, *are found to differ ſenſibly from* thoſe *computed on the hypotheſis of a parabolic orbit.*

The Fourth, *is an attempt to ſhew, from mathematical principles, the advantage ariſing by taking the* mean *of a* number *of obſervations, in* practical Aſtronomy; *wherein the odds that the reſult, this way, is more exact, than from one ſingle obſervation, is evinced, and the utility of the method in practice, clearly made to appear.—A part of* this, *and of the 7th paper, is inſerted in the* XLIX *volume of the* Philoſophical Tranſactions; *but the farther improvements here added, will (I hope) be a ſufficient apology for my printing the whole again, in this* work.

The Fifth, *contains the determination of certain* Fluents, *and the reſolution of ſome very uſeful* Equations, *in the higher orders of fluxions, by means of the meaſures of angles and ratios, and the right ſines, and verſed ſines of circular arcs.*

The Sixth, *treats of the reſolution of algebraical equations, by the method of ſurd-diviſors; wherein the grounds of that method, as laid down by* Sir Iſaac Newton, *are inveſtigated and explained.*

The Seventh, *exhibits the inveſtigation of a* general rule *for the reſolution of* Iſoperimetrical Problems *of all orders, together with ſome examples of the uſe and application of the ſaid rule.*

The Eighth (and laſt) Part, *comprehends the reſolution of ſome general, and very intereſting problems, in* mechanics *and* phyſical Aſtronomy; *wherein, among other particulars, the principal parts of the third, and ninth* ſections *of the* firſt Book *of Sir* Iſaac Newton's Principia, *are demonſtrated, in a new, and very conciſe manner.—But what, I apprehend, may beſt recommend this part of the work, is the application of the general equations therein derived, to the determination of the* lunar Orbit: *In which I have exerted my utmoſt endeavours to render the whole intelligible even to Thoſe who have arrived but to a tolerable proficiency in the higher geometry.*

The greater part of what is here delivered on this ſubject, was drawn up in the year 1750, *agreeably to what is intimated at the concluſion of my* Doctrine of Fluxions, *where the general equations are alſo given. The famous objection, about that time made to* Sir Iſaac Newton's *general Law of Gravitation, by that eminent mathematician* M. Clairaut, *of the* Royal Academy of Sciences *at* PARIS, *was a motive ſufficient to induce* me *(among many* Others) *to endeavour to diſcover, whether the motion of the moon's* apogee, *on which that objection had its whole weight and foundation, could not be truly accounted for, without ſuppoſing a change in the received law of gravitation, from the inverſe ratio of the ſquares of the diſtances. The ſucceſs was anſwerable to my hopes, and ſuch as induced me to look farther into other parts of the theory of the moon's motion, than I firſt intended: but, before I had completed my deſign, I received the honour of a viſit from* M. Clairaut *(juſt then arrived in* England) *of whom I learned, that he had a little before printed a* piece *on that ſubject; a copy of which I afterwards received, as a preſent at his hands; wherein I found moſt of the ſame things demonſtrated, beſides ſeveral others, to which I had not then extended my enquiry. Upon this, I at that time deſiſted from a farther proſecution of the ſubject; being chiefly diverted therefrom by a call then ſubſiſting for a new edition of another work, in which ſome* additions *ſeemed wanting. But I cannot omit to obſerve here, in juſtice to* M. Clairaut, *that, tho' he indeed fell into a miſtake, by too haſtily inferring a defect in the received law of attraction, from the inſufficiency of the known methods for determining the effect of that attraction, in the motion of the moon's* apogee, *yet he was himſelf, the firſt who diſcovered the true ſource, of that miſtake, and who placed the matter in a proper light: Though there are* ſome[*] *who have, both before and ſince, undertaken to give the true quantity of that motion, from ſuch principles, only, as are laid down in the ninth* ſection *of the firſt* Book *of the* Principia: *but that theſe* Gentlemen, *however they may have made their numbers to agree, have been greatly deceived in their calculations, is very certain; ſince a conſiderable part of the ſaid motion depends on that part of the* ſolar force *acting in the direction perpendicular to the* Radius-vector, *which is by them, either intirely diſregarded, or the effect thereof, not made one twentieth part of what it really ought to be.—There are Others indeed, who have explained the matter, upon true principles, and with better ſucceſs. Since* M. Clairaut*'s piece firſt made its appearance, the moſt eminent* mathematicians, *in different parts of* Europe, *have turned their thoughts that way. But tho' what I now offer on the ſame ſubject, may, perhaps, appear of leſs value, after what has been already done by theſe great men, yet I am not very ſolicitous upon that account, as it will be found, that I have neither copied from*

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[*] *Vid. Walmſley's Theorie du mouvement des apſides* (tranſlated into Engliſh) *and Vol.* 47. N° 11. *of the Philoſophical Tranſactions.*

their thoughts, nor detracted from their merit. The facility *of the method I have fallen upon, will, I flatter myſelf, be allowed by* all, *who are appriz'd of the real difficulty of the ſubject; and the* extenſiveneſs *thereof will, in ſome meaſure, appear from this, that* it *not only determines the motion of the* apogee *in the ſame manner, and with the ſame eaſe, as the other equations, but utterly excludes, at the ſame time, all terms of that dangerous ſpecies (if I may ſo expreſs myſelf) that have hitherto embarraſſed the greateſt* Mathematicians, *and that would, after a great number of revolutions, intirely change the figure of the orbit. It* thereby *appears, that all the terms, or equations in general, will be expreſſed by ſines and co-ſines, barely, without any multiplication into the arcs correſponding. From whence this important conſequence is derived, that the mean motion, and the greateſt quantities of the ſeveral equations will remain unchanged; unleſs diſturbed by the intervention of ſome foreign, or accidental cauſe.*

In treating of this ſubject, as well as in moſt of the other parts of the enſuing work, I have chiefly adhered to the analytic method of Inveſtigation, *as being the moſt direct and extenſive, and beſt adapted to theſe abſtruſe kinds of ſpeculations. Where a* geometrical demonſtration *could be introduced, and ſeemed preferable, I have given one: but, tho' a problem, ſometimes, by this laſt method, acquires a degree of perſpicuity and elegance, not eaſy to be arrived at any other way, yet I cannot be of the opinion of Thoſe who affect to ſhew a diſlike to every thing performed by means of* ſymbols *and an* algebraical Proceſs; *ſince, ſo far is the* ſynthetic method *from having the advantage in all caſes, that there are innumerable enquiries into nature, as well as in abſtracted ſcience, where it cannot be at all applied, to any purpoſe.* Sir Iſaac Newton *himſelf (who perhaps extended* it *as far as any* man *could) has even in the moſt ſimple caſe of the lunar orbit* (Princip. *B.* 3. *prop.* 28*) been obliged to call in the aſſiſtance of* algebra; *which he has alſo done, in treating of the motion of bodies in reſiſting mediums, and in various other places. And it appears clear to me, that, it is by a diligent cultivation of the* Modern Analyſis, *that* Foreign Mathematicians *have, of late, been able to puſh their* Reſearches *farther, in many particulars, than* Sir Iſaac Newton *and his* Followers *here, have done: tho' it muſt be allowed, on the other hand, that the ſame* Neatneſs, *and* Accuracy of Demonſtration, *is not every-where to be found in thoſe* Authors; *owing in ſome meaſure, perhaps, to too great a diſregard for the* Geometry of the Antients.

A
DETERMINATION
OF THE
PRECESSION OF THE EQUINOX,
And the different Motions of the Earth's Axis,
Ariſing from the Attraction of the Sun and Moon.

THE Precession of the *Equinox*, whereby the fix'd ſtars appear to have changed their places by more than a whole *ſign*, ſince the time of the moſt ancient *Aſtronomers*, is phyſically accounted for, from the attraction of the *ſun* and *moon* on the protuberant matter about the earth's equator; whereby the poſition of the ſaid equator with reſpect to the plane of the ecliptic is ſubjected to a perpetual variation. Were the earth to be perfectly ſpherical and of an uniform denſity, no change in the poſition of the terreſtrial equator could be produced, from the attraction of any remote body; becauſe the force of each particle of matter in the earth, to turn the whole earth about its center, in conſequence of ſuch attraction, would then be exactly counterbalanced by an equal, and contrary force. But as the earth, by reaſon of the centrifugal force of the parts thereof, ariſing from the diurnal rotation, muſt, to preſerve an *equilibrium*, put on an oblate figure, and riſe higher about the equatoreal parts than at the poles, the action of the ſun on the ſaid equatoreal parts will have an effect to make the plane of the terreſtrial equator to coincide with that of the ecliptic: which would actually be brought to paſs (neglecting other cauſes) was the ſun, or earth, to remain fix'd in either of the *ſolſtices*, and the diurnal rotation at the ſame time to ceaſe. But, though both the motions of the earth contribute to prevent an effect of that ſort, yet, in conſequence of this action of the ſun, a new motion of rotation, about that diameter of the equator lying in the circle of the ſun's declination, is produced; from which the preceſſion of the equinox and the nutation of the earth's axis have their riſe.

The effect of the moon, as it is much more conſiderable than that of the ſun, ſo is it likewiſe liable to ſome inequalities to which that of the ſun is not ſubject. Were the inclination of the lunar orbit to the plane of the equator to remain, always, nearly the ſame, like that of the earth, the ſame calculations that anſwer'd in the one caſe would alſo anſwer in the other; but that inclination is continually varying, and, when the aſcending node is in the beginning of *Aries*, is greater by above $\frac{1}{6}$th part than the mean value; and therefore, as the force of the moon to turn the earth about its center (other circumſtances remaining the ſame) is found, hereafter, to be as the ſine of the double of the inclination, it is manifeſt, that, in the ſaid poſition of the node, the motion of preceſſion will go on much quicker than at the mean rate; and conſequently that an equation, depending on the place of the node, will neceſſarily ariſe. The determination of which, as well as of the other motions of preceſſion and nutation ariſing from the attraction both of the ſun and moon, I ſhall now proceed to ſhew: but in order to pave the way thereto, it will be proper to begin with premiſing the ſubſequent *Lemmas.*

LEMMA I.

Suppoſing all the particles of a given ſpheroid A′P*áp*O *to be ſollicited parallel to the axis* P*p*, *by forces proportional to the diſtances from a plane* PAO*pa paſſing by the ſaid axis, in ſuch ſort that the two oppoſite ſemi-ſpheroids,* A′P*p*, *á*P*p*, *may thereby be equally urged in contrary directions; it is propoſed to determine the whole effect of all the forces to turn the ſpheroid about its center.*

Let $\left\{ \begin{array}{l} a = \text{ſemi-diam. OA}' \text{ (perpend. to the plane PAO}pa\text{)}, \\ A = \text{area of the ellipſe PAO}pa, \\ \gamma = \text{force acting on a particle at the remoteſt point A}', \\ x = \text{ON, the diſt. of any ſection DQNEQ from PAO}pa\text{:} \end{array} \right.$ Fig. 1.

Then, this ſection being alſo an ellipſe, ſimilar to PA*pa*, we ſhall have, by the property of the ellipſis, as $A'O^2\,(aa) : A'O^2 - ON^2\,(aa - xx) :: PO^2 : DN^2 ::$ the area PAO*pa* (A) to the area DQNEQ $= A \times \frac{aa - xx}{aa}$ (by the property of ſimilar figures). Hence it is evident that $A \times \frac{aa - xx}{aa} \times \frac{x}{a} \times \gamma$ will be the ſum of all the forces whereby the particles in the ellipſe DQEQ are urged parallel to the axis P*p* of the ſpheroid; which quantity, drawn into (x) the length of the lever ON, will, conſequently, expreſs the effect of all the ſaid forces to turn the ſpheroid about its center: and ſo the fluent of $A \times \frac{aa - xx}{aa} \times \frac{x}{a} \times x\dot{x} \times \gamma$, which is $A \times \frac{2aa}{15} \times \gamma$ (when $x = a$) will truly expreſs one half of the quantity ſought.

COROLLARY.

If the maſs, or content of the ſpheroid, which is $A \times 2a \times \frac{2}{3}$, be denoted by S; then the force $A \times \frac{4aa}{15} \times \gamma$, whereby the ſpheroid tends to turn about its center, will be truly defined by $\frac{1}{5} S \times a \times \gamma$, which therefore is juſt $\frac{1}{5}$th part of what it would be, if all the particles were to act at the diſtance of the remoteſt point A′.

LEMMA II.

Suppoſe a body to revolve in the circumference of a circle AF*a*F, Fig. 2. *whilſt the circle itſelf turns uniformly about one of its diameters* A*a*, *as an axis, with a very ſlow motion; it is propoſed to determine the law of the force, acting on the body in a direction perpendicular to the plane of the circle, neceſſary to the continuation of a motion thus compounded.*

Fig. 2. Let $\mathrm{AF}a\mathrm{F}$ and $\mathrm{A}faf$ be two poſitions of the circle, indefinitely near to each other, and let R and r be the two correſponding poſitions of the body; let alſo the planes $\mathrm{RD}n$ and mdc be perpendicular to $\mathrm{AF}a\mathrm{F}$ and to the axis $\mathrm{AO}a$; in which planes let there be drawn $\mathrm{R}n$ and mvc perpendicular to DR and dm, meeting the plane $\mathrm{A}raf$ (produced out) in n and c; and let there be drawn nv, parallel to the tangent $\mathrm{R}tm$, meeting mc in v. If the velocity of the body along the circumference be expreſſed by $\mathrm{R}m$, the velocity in the perpendicular direction $\mathrm{R}n$, ariſing from the motion of the circle about the axis $\mathrm{A}a$, will be repreſented by $\mathrm{R}n$. And, if the body were to be ſuffered to purſue its own direction from the point R, it would, by the compoſition of thoſe motions, arrive at the oppoſite angle v of the parallelogram $\mathrm{R}nvm$, in the ſame time that it might move through $\mathrm{R}m$ by the motion $\mathrm{R}m$ alone; and ſo would fall ſhort of the plane by the diſtance cv*. It therefore appears that the required force, neceſſary to keep the body in the plane, muſt be ſuch as is ſufficient to cauſe a body to move over the diſtance cv in the aforeſaid time; and that *this force* muſt, therefore, be to the centrifugal force of the body in the circumference (whoſe meaſure is et) as cv to et; ſince the ſpaces deſcribed in equal times, are directly as the accelerating forces.

Let now the ratio of the angular celerity of the circle about its axis to *that* of the body in the circumference, be ſuppoſed as r to *unity*; then, the latter of theſe celerities being repreſented by $\mathrm{R}m$, the former will be defined by $r\times\mathrm{R}m$; and conſequently the celerity $(\mathrm{R}n)$ in the direction $\mathrm{R}n$, by $r\times\mathrm{R}m\times\frac{\mathrm{DR}}{\mathrm{OF}}$. Moreover, becauſe of the ſimilarity of the triangles $\mathrm{DR}n$ and dmc, it will be, as $\mathrm{DR} : \mathrm{R}n\ \left(r\times\mathrm{R}m\times\frac{\mathrm{DR}}{\mathrm{OF}}\right) :: dm\,(\mathrm{DR}+sm) : mc = r\times\frac{\mathrm{R}m\times\mathrm{DR}}{\mathrm{OF}} + r\times\frac{\mathrm{R}m\times sm}{\mathrm{OF}}$: from whence, taking away the value of mv or $\mathrm{R}n$, we get $cv = r\times\frac{\mathrm{R}m\times sm}{\mathrm{OF}}$: which is in proportion to the meaſure of the centrifugal force *et*, or it's equal $\frac{\overline{Rm}^2}{2OR}$, as $r \times sm$ to $\frac{1}{2}Rm$, or, becauſe of the ſimilar triangles ORD and R*sm*, as $2r \times OD$ to OR or OA.

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* The lineola cr, lying in the plane of the circle, muſt be anſwered by a force tending to the center of the circle; with which we have nothing to do in the preſent conſideration.

Hence it is evident that the body, to continue in the plane of the circle, muſt be *conſtantly* acted on, in a direction perpendicular to the plane, by a force varying according to the coſine of the diſtance AR of the body from the extremity of the axis; whoſe greateſt value, at A, is to the centrifugal force in the circle, as $2r$ to unity. *Q. E. I.*

COROLLARY I.

If, inſtead of one, a great number of bodies or corpuſcles, ſo as to touch one another and thereby form a continued ring AF*a*F, were to revolve at the ſame time, and to be acted on in the ſame manner (that is to ſay, by forces in the ratio of the diſtances from the diameter FF perpendicular to the axis A*a*), it is evident that they would all continue in the ſame plane. And this will alſo be the caſe, when a number of concentric rings ERG*e*G, &c. are ſuppoſed to perform their revolutions together about the common axis AE*ea*. For, aſſuming β to denote the centrifugal force of a corpuſcle in the outermoſt ring AR′F*a*F, the centrifugal force of an equal corpuſcle (R′) in the ring ER′*e*G, will be equal to $\beta \times \frac{OE}{OA}$: whence, by the foregoing proportions, $2r \times \beta \times \frac{OE}{OA}$ will be the force acting perpendicular to the plane at E: and $2r \times \beta \times \frac{OE}{OA} \times \frac{OD}{OR'}$ $\left(= 2r \times \beta \times \frac{OD}{OA}\right)$ will be the true meaſure of the force acting on a corpuſcle at R′; which, as r, β, and OA are all of them conſtant, is evidently as the diſtance from the diameter FF. Whence it follows, becauſe the diſtance below FF becomes negative, that the forces above and below that diameter muſt have contrary directions.

Fig. 3.

COROLLARY II.

Whatever hath been ſaid in the preceding Corollary holds equally, when the line or axis A*a*, about which the plane is ſuppoſed to turn, hath a progreſſive motion, or is carried uniformly forward, parallel to itſelf; provided the angular celerity about that axis continues the ſame; as is evident from the reſolution of forces. Fig. 4. Hence it follows, that, if a circle E′EC*ée*, conſidered as compoſed of an indefinite number of concentric rings, be ſuppoſed to revolve uniformly about its center C, whilſt the center itſelf and the right-line OC (which, to help the imagination, may be taken as the axis of a cone E′O*é*, whoſe baſe is E′E*eé*) move uniformly in the plane P*áp*A′ about the point O; I ſay, it follows that the forces neceſſary to keep the particles in the plane, under ſuch a compound motion, will be the very ſame as if the circle was to turn about the line E*e* (perpendicular to the plane P*áp*A′) at reſt, with an angular celerity equal to *that* of the center C about the point O: becauſe, the angle OCE′ being *always* a right one, the angular celerity of the moveable circle about the line EC*e* (which remains every-where parallel to itſelf) will, evidently, be equal to the angular celerity of the center of the circle about the point O. From whence and the preceding Corollary it is manifeſt, that the forces which, acting parallel to PCO, are neceſſary to retain the particles in the plane E′E*ée*, will be, every-where, as the diſtances from the diameter E′C*é*, or the plane P*áp*A′, let the diſtance of the plane E′E*ée* from the center O be what it will.

COROLLARY III.

Conceive now OAP*ap*A′*á* to be an homogenous fluid, revolving uniformly about the axis PO*p*, under the form of an oblate ſpheroid *; whilſt the axis itſelf is ſuppoſed to turn about the center O, in the manner explained above: then it will appear, from what is there delivered, that the particles of the fluid, to continue in *equilibrio* among themſelves, muſt be ſolicited parallel to the axis, by forces that are as the diſtances from the plane P*áp*A′; ſuch, that the force acting at the remoteſt point A may be defined by $2r\beta$; where β (*by Corol. I.*)

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* That the particles will remain in equilibrio, under the form of an oblate ſpheroid (when the axis is at reſt), is demonſtrated in Part II. Sect. 9. of my *Doctrine and Application of Fluxions.*

repreſents the centrifugal force in the circumference A*aa*A′ of the greateſt circle, and *r* the meaſure of the angular motion of the axis itſelf, that of the rotation, about the axis, being denoted by *unity*. But it appears further, *from Lemma I*, that the efficacy of all the ſaid forces to turn the ſpheroid about its center (making γ here $= 2r\beta$) is truly defined by $2r\beta \times \frac{1}{5}S \times OA$. Whence it is plain, that all the particles of the body will remain *in equilibrio* among themſelves, under the two different motions above explained, when the *whole* force producing the motion of the axis, is expreſſed by $2r\beta \times \frac{1}{5}S \times OA$. And, when the forces reſpecting the ſeveral particles are ſuppoſed to act according to a different law, the effect produced by them will be the ſame, provided their *joint efficacy*, to turn the body about its center, *be the ſame:* ſince the ſame force muſt be anſwered, or ſatisfied with the ſame kind and degree of motion in the whole body; if we except only, the exceeding ſmall difference that will ariſe from the alteration of the figure; which figure will not be *accurately* a ſpheroid, in this caſe, but *nearly* ſuch, as the motion of the axis and, conſequently, the forces producing it, are ſuppoſed very ſmall. Neither will the axis continue to move in the ſame plane, when the direction of the forces is not every-where parallel to the axis; the motion produced in the body being always about that diameter (A*a*) wherein the whole perturbating force may be conceived to act, as by a lever, to turn the body about its center. Laſtly, it may be obſerved here, that the time of revolution about the axis will not, in this caſe, continue *accurately* the ſame; ſince a change of the figure muſt neceſſarily be attended with a change in the time of revolution. But this change of motion about the axis, when we regard the effect of the perturbating forces of the ſun and moon upon the earth, is ſo extremely ſmall, as to be quite inconſiderable, even in compariſon of the very ſlow motion of the axis above ſpoken of. Fig. 4.

LEMMA III.

Suppoſing all the particles of a given ellipſe MFN*f to be urged from a right-line* GG *coinciding with a given diameter* MN, *by forces*

forces proportional to the diſtances from the ſaid line, ſuch that the force acting at a given diſtance a, may be expreſſed by a given quantity γ; it is required to find the whole efficacy of all theſe forces, to turn the ellipſe about its center O.

If BC be ſuppoſed parallel to GG, interſecting OT, perpendicular thereto, in D; then the force with which a particle, at any place V in that line, is urged in the direction *w*V parallel to OD, will be expreſſed by $\frac{\gamma}{a} \times Vw$, or $\frac{\gamma}{a} \times OD$; and conſequently it's efficacy to turn the ellipſe about its center by $\frac{\gamma}{a} \times OD \times Ow$, or $\frac{\gamma}{a} \times OD \times DV$. Let there be taken $Cv =$ DV; and the efficacy of a particle at *v* will, in like manner, be had equal to $\frac{\gamma}{a} \times OD \times Dv$: which, added to that of the
Fig. 5. former particle at V, gives $\frac{\gamma}{a} \times OD \times DC$. Therefore, ſeeing the joint action of any two particles in DC, equally diſtant from the middle one I, is expreſſed by the *ſame* quantity $\frac{\gamma}{a} \times OD \times DC$, the efficacy of all the particles muſt conſequently be equal to that quantity drawn into half the number of the particles; and ſo is truly expounded by $\frac{\gamma}{a} \times \frac{1}{2} OD \times DC^2$. By the ſame argument, the force of all the particles in the line BD to turn the ellipſe about its center, the contrary way, will be $\frac{\gamma}{a} \times \frac{1}{2} OD \times BD^2$. Therefore the difference of theſe two values, $\frac{\gamma}{a} \times \frac{1}{2} OD \times \overline{BD^2 - CD^2}$, is the whole force of all the particles in the line BC, to turn the ellipſe about its center (downwards); which expreſſion, if F*f* the conjugate diameter to MN be drawn, biſecting BC in E, will become $\frac{\gamma}{a} \times \frac{1}{2} OD \times \overline{BD + CD} \times \overline{BD - CD} = \frac{\gamma}{a} \times OD \times BC \times DE$.

Put, now, $OF = c$, $OM = d$, FH (perpendicular to MN) $= f$, $OH = g$; and let OE and OD, conſidered as variable, be denoted by x and y, reſpectively. Then, by the property of the ellipſis, it will be, $cc : dd :: cc - xx : \overline{\mathrm{BE}}\rvert^{2} = \frac{dd \times \overline{cc - xx}}{cc}$; and conſequently $\mathrm{BC} = \frac{2d\sqrt{cc-xx}}{c}$. Alſo (*by ſimilar triangles*) $c : f :: x : y = \frac{fx}{c}$; and $c : g :: x : \mathrm{DE} = \frac{gx}{c}$. Hence our expreſſion $\frac{\gamma}{a} \times \mathrm{OD} \times \mathrm{DE} \times \mathrm{BC}$, derived above, by ſubſtituting theſe values, becomes $\frac{\gamma}{a} \times \frac{2dfg}{c^3} \times \overline{cc - xx}\rvert^{\frac{1}{2}} \times x^2$: and therefore the whole fluent of $\frac{\gamma}{a} \times \frac{2dfg}{c^3} \times \overline{cc - xx}\rvert^{\frac{1}{2}} \times xx\dot{y}$, or of its equal $\frac{\gamma}{a} \times \frac{2dfg}{c^3} \times \overline{cc - xx}\rvert^{\frac{1}{2}} \times x^2 \times \frac{f\dot{x}}{c}$, will be the force of all the particles in the ſemi-ellipſe MFN. In order to the finding of this fluent, let *A* be taken to denote the area of the ſemi-ellipſe, or, which is the ſame, the fluent of $\frac{2d}{c} \times \overline{cc - xx}\rvert^{\frac{1}{2}} \times \frac{f\dot{x}}{c}$; then, by compariſon, the *whole* fluent of $\frac{2d}{c} \times \overline{cc - xx}\rvert^{\frac{1}{2}} \times x^2 \times \frac{f\dot{x}}{c}$, when $x = c$, will be found to be $A \times \frac{1}{4}c^2$: whence *that* of our given expreſſion, $\frac{\gamma}{a} \times \frac{2dfg}{c^3} \times \overline{cc - xx}\rvert^{\frac{1}{2}} \times x^2 \times \frac{f\dot{x}}{c}$, muſt conſequently be $= \frac{\gamma}{a} \times \frac{fg}{cc} \times \frac{1}{4}c^2\mathrm{A} = \frac{\gamma}{a} \times \frac{1}{4}fg\mathrm{A} = \frac{\gamma}{a} \times \frac{1}{4}\mathrm{FH} \times \mathrm{OH} \times A$; the double of which, or $\frac{\gamma}{a} \times \frac{1}{4}\mathrm{FH} \times \mathrm{OH} \times$ *area* MFN*f*M, is therefore the true meaſure of the whole force whereby the ellipſe tends to move about its center. *Q. E. I.*

COROLLARY I.

If the ſame value be required by means of the angle AOM included between the diameter MN and the principal axis AO*a* (ſuppoſed to be given); then let PO*p* and FR be drawn perpendicular to OA, and TF to OT, meeting OA produced, in Q; ſuppoſe L to be the interſection of AO and FH; and let Fig. 6. the ſine and co-ſine of the ſaid given angle AOM (to the radius 1) be denoted by *m* and *n*, reſpectively. Becauſe FL is perpendicular to the tangent TQ, we have, by the property of the ellipſis, as $AO^2 : AO^2 - OP^2 :: OR : OL :: OR \times OQ$ $(AO^2) : OL \times OQ$; and conſequently $AO^2 - OP^2 = OL \times OQ$.

But, $1 : n :: OL : OH$;
and, $1 : m :: OQ : OT\ (FH)$;

whence, by compoſition, $1 : mn :: OL \times OQ\ (= AO^2 - OP^2) : FH \times OH = mn \times \overline{AO^2 - OP^2}$: and ſo, by ſubſtituting this value above, we get $\frac{\gamma}{a} \times \frac{mn}{4} \times \overline{AO^2 - OP^2} \times$ *area of the ellipſe,* for another expreſſion of the required force.

COROLLARY II.

Hence may be eaſily deduced the force by which the ſpheroid, generated by the rotation of the ellipſe about its leſſer axe *Pp,* tends to turn about its center, when all the particles are urged from a plane GG paſſing through the center, by forces proportional to the diſtances from the ſaid plane. For, as any ſection of the ſpheroid, parallel to the middle one A*pa*P, is alſo an ellipſe, ſimilar to it, the area of that ſection will be in proportion to the area of A*pa*P (which I ſhall denote by $\mathcal{Q}$) as the ſquare of its greater ſemi-axe, to the ſquare of the greater ſemi-axe OA of the given ellipſe PA*pa*: ſo that, if OA be denoted by a, PO by b, and the diſtance of the ſaid ſection from the center of the ſpheroid by u, we ſhall have, $aa : aa - uu$ (= ſq. greater ſemi-axis of that ſection, by the property of the circle) $:: \mathcal{Q} : \mathcal{Q} \times \frac{aa - uu}{aa}$, the area of the ſection. Moreover, by reaſon of the ſimilar figures, we have $aa : aa - bb :: aa - uu : \frac{aa - bb}{aa} \times \overline{aa - uu}$, the difference of the ſquares of the greater and leſſer ſemi-axes of the ſection. Therefore, by ſubſtituting theſe values in the above general expreſſion, we get $\frac{\gamma}{a} \times \frac{mn}{4} \times \frac{aa - bb}{aa} \times \overline{aa - uu} \times \mathcal{Q} \times \frac{aa - uu}{aa}$ $\left(= \frac{\gamma}{a} \times \frac{mn}{4} \times \frac{aa - bb}{aa} \times \mathcal{Q} \times \frac{a^4 - 2a^2u^2 + u^4}{aa}\right)$ for the force of all the particles in that ſection to turn the body about the common axis of motion ſtanding at right-angles to the plane PA*pa*. This quantity, drawn into $\dot{u}$, will, therefore, be the fluxion of force of the ſemi-ſpheroid in which that ſection is; whoſe fluent, when $u = a$, will be found $= \frac{\gamma}{a} \times \frac{mn}{4} \times \frac{aa - bb}{aa} \times Q \times \frac{8a^3}{15}$: the double whereof, or $\frac{\gamma}{a} \times mn \times \overline{aa - bb} \times \frac{4aQ}{15}$, muſt conſequently be the required force of the whole ſpheroid: which force, as $Q \times \frac{4a}{3}$ is known to expreſs the content, or maſs of the ſpheroid, will alſo be truly defined by $\frac{\gamma}{a} \times \frac{mn}{5} \times \overline{aa - bb} \times S$; S being put (as in the preceding Lemmas) to repreſent the ſaid content or maſs.

PROBLEM I.

To determine the efficacy of the ſun's attraction, on a corpuſcle, any where in the body of the earth, to turn the earth about it's center.

Let CDHE repreſent the earth, C the center thereof, S that of the ſun, and GCG a plane perpendicular to the line CS joining the centers of the earth and ſun; let D be the place of the corpuſcle, and upon the diagonal SD let the parallelogram QCSD be conſtituted; producing QD to meet GCG in K. Fig. 7.

If F be taken to denote the ſun's abſolute force on a particle at the center C, his force on a particle at D will be $F \times \frac{SC^2}{SD^2}$; which may be reſolved into two others, the one in the direction DC (which has no effect at all to turn the earth about its center); and the other in the direction DQ, expreſſed by $F \times \frac{SC^2}{SD^2} \times \frac{DQ}{SD}$: from which the force F, in the parallel direction CS, being deducted, the remainder $F \times \frac{SC^3 - SD^3}{SD^3}$ will be the true meaſure of that part of the force in the direction DQ, whereby the particle at D tends to change its poſition with reſpect to the plane GCG. But this value is reducible to $F \times \frac{\overline{SC - SD} \times \overline{SC^2 + SC \times SD + SD^2}}{SD^3}$; which, as SC — SD (by reaſon of the great diſtance of the ſun) is nearly equal to

DK, will become $= F\times\frac{\text{DK}\times 3\text{SD}^2}{\text{SD}^3} = 3F\times\frac{\text{DK}}{\text{SC}}$, nearly: which, drawn into CK, will be as the required efficacy of that force to turn the body about its center. *Q. E. I.*

COROLLARY I.

If there be taken
$A=$ (SC) the diſtance of the ſun and earth,
$a=$ the ſemi-equatoreal diameter of the earth,
$T=$ the time of the annual revolution,
$t=$ the time of the diurnal revolution, and
$\beta=$ the centrifugal force of a particle at the equator, ariſing from the diurnal revolution;

then, ſince $\frac{a}{tt} : \frac{A}{TT} :: \beta : F$, or $F=\frac{A\beta tt}{aTT}$ (by the known laws of central forces) it is evident that the force $3F\times\frac{\text{DK}}{\text{SC}}$, with which a particle at D tends from the plane GCG, will alſo be truly defined by $\beta\times\frac{3tt}{TT}\times\frac{\text{DK}}{a}$. Hence it appears that the ſaid force is directly as the diſtance from the plane; and that the value thereof, at the diſtance of the earth's ſemi-equatoreal diameter, is truly defined by $\beta\times\frac{3tt}{TT}$; being in proportion to the centrifugal force at the equator, ariſing from the diurnal rotation, as thrice the ſquare of the time of the diurnal revolution, to once the ſquare of the time of the annual revolution.

COROLLARY II.

Fig. 7. Moreover, from hence the perturbating force of the moon, or any other planet, will be given: for, ſuppoſing S to repreſent the planet, it's abſolute force (F) at the center C, will be as its quantity of matter, applied to the ſquare of the diſtance SC: and ſo our general expreſſion $3F\times\frac{\text{DK}}{\text{CS}}$ will here become $\frac{3\text{DK}\times S}{\overline{\text{SC}}|^{3}}$; which, becauſe the quantities of matter in bodies of the ſame denſity, are as the cubes of their ſemi-diameters, will alſo (ſuppoſing the poſition of D to remain the ſame)

be as the cube of the ſemi-diameter of the planet directly, and the cube of its diſtance SC, inverſly; or, which is the ſame, as the cube of the ſine of the apparent ſemi-diameter directly, and the cube of the radius inverſly. Hence it is manifeſt, that the perturbating forces of planets, of the ſame denſity, are in proportion directly as the cubes of the ſines of their apparent ſemi-diameters, or as the cubes of the ſemi-diameters, themſelves, very near. Therefore the ſun and moon, appearing under equal ſemi-diameters, have their perturbating forces in the ſame ratio with their denſities.

PROBLEM II.

To determine the change of the poſition of the terreſtrial equator, ariſing from the action of the ſun on the whole maſs of the earth, during any very ſmall interval of time.

Fig. 8.

Let OAP*ap* be the earth, under the form of an oblate ſpheroid; let AI*a*L be the plane of its equator, and HICL a plane paſſing through S the center of the ſun, and making right-angles with the plane of the meridian HAPC*p*.

It appears, *by Corol. I. to Prop. I.* that the relative force whereby a particle of matter, any where in the earth, tends, through the action of the ſun, to recede from the plane GG, perpendicular to HICL, is directly as the diſtance from the ſaid plane; and that the ſaid force, at the diſtance (a) of the ſemi-equatoreal diameter from the plane, is truly defined by $\beta \times \frac{3tt}{TT}$.

It appears moreover, *from Corol. II. to Lem. III.* that a ſpheroid, acted on in this manner, tends, through the joint force of all the particles, to turn about its center with a force expreſſed by $\frac{\gamma}{a} \times \frac{mn \times \overline{aa - bb}}{5} \times S$; γ being the force wherewith a particle, at the diſtance a, is urged from the plane GG. Therefore, expounding γ by $\beta \times \frac{3tt}{TT}$, we have $\beta \times \frac{3tt}{TT} \times \frac{mn \times \overline{aa - bb}}{5a} \times S$, for the true meaſure of the force whereby the whole earth tends to turn about its center, through the ſun's attraction.

But it is proved, *in Corol. III. to Lem. II.* that, if a ſpheroid OAP*ap* revolving about its axis P*p*, be at the ſame time acted on by forces tending to generate a new motion, at right angles to the former, it will, in conſequence of ſuch action, have another motion, about the line A*a*; whereof the celerity will be in proportion to *that* of the former motion about the axis (P*p*), as r to 1; the whole force with which the ſpheroid tends to turn about its center, whereby this motion is produced, being expreſſed by $2r\beta \times \frac{1}{5}S \times a$. Let this force, therefore, be made equal to *that* found above, by which the earth tends to turn about its center: by which means we have $\frac{2ra}{5} = \frac{3tt}{TT} \times \frac{mn \times \overline{aa - bb}}{5a}$; and therefore $r = \frac{3tt}{2TT} \times \frac{mn \times \overline{aa - bb}}{aa}$. From whence it appears that the earth, in conſequence of the ſun's attraction, has a motion about the line A*a* (lying in the plane of the ſun's declination) whereof the celerity will be in proportion to *that* of the diurnal motion about the axis P*p*, as $\frac{3tt}{2TT} \times \frac{mn \times \overline{aa - bb}}{aa}$ to *unity*: where t and T expreſs the reſpective times of the diurnal, and annual revolutions, a and b the greateſt, and leaſt ſemi-diameters of the earth, and m and n the ſines of the ſun's declination and polar diſtance.

PROBLEM III.

To determine the preceſſion of the equinox, and the nutation of the earth's axis, cauſed by the ſun, during any very ſmall interval of time; on the ſuppoſition of an uniform denſity of all the parts of the earth.

Fig. 9. Let ≏D♈ be the ecliptic, on the ſurface of the ſphere; and let ≏AC♈ be the poſition of the equator, when S is the ſun's place in the ecliptic, and SA his declination.

It is evident, from the laſt propoſition, that the angle ≏A*a*, or ♈A*b*, deſcribed by the equator about the point A, in any very ſmall time t', by means of the ſun's attraction, is in proportion to $\left(\frac{t'}{t} \times 360^\circ\right)$ the angle deſcribed in the ſame time, by means of the diurnal motion, as $\frac{3tt}{2TT} \times \frac{mn \times \overline{aa - bb}}{aa}$ is to unity; and therefore is truly defined by $360^\circ \times \frac{3tt'}{2TT} \times \frac{mn \times \overline{aa - bb}}{aa}$, or $360^\circ \times \frac{3tt'}{2TT} \times kmn$; ſuppoſing $\frac{aa - bb}{aa}$ (for the ſake of brevity) to be denoted by k. But it will be *(p. ſpherics)* as ſine of a (or ♎) : ſin. ♎A (: : ſin. ♎Aa : ſin. ♎a) : : angle ♎Aa : ♎$a = 360^\circ \times \frac{3tt'}{2TT} \times kmn \times \frac{\text{ſin. ♎A}}{\text{ſin. ♎}}$, the quantity of the preceſſion required.

Again, if aD and ac be taken as arcs of 90 degrees each; ſo that Dc may be the meaſure of the angle a; we ſhall alſo have *(p. ſpherics)* as Rad. : ſin. AC (co-ſin. ♎A) : : ſin. CAc (♎Aa) : ſin. Cc : : angle ♎Aa : C$c = 360^\circ \times \frac{3tt'}{2TT} \times kmn \times \frac{\text{co-ſ. ♎A}}{\textit{Rad.}}$, the required nutation, or the decreaſe of the inclination of the equator to the ecliptic. *Q. E. I.*

COROLLARY.

If ♎e be made perpendicular to aA, it will be, as ♎e : Cc (: : ſin. ♎A : ſin. CA) : : tang. ♎A : *Radius*;

alſo, ♎a : ♎e : : Rad. : ſin. a : whence, by compounding theſe proportions, we have ♎a : Cc : : tang. ♎A : ſin. a. From which it appears, that the quantity of the preceſſion, for any very ſmall time, is to that of the nutation correſponding, as the tangent of the ſun's right aſcenſion to the ſine of the inclination of the equator to the ecliptic.

PROBLEM IV.

To determine the preceſſion of the equinox, and the nutation of the earth's axis, cauſed by the ſun, from the time of his appearing in the equinoctial point, to his arrival at any given diſtance therefrom.

Every thing being ſuppoſed as in the preceding Problem, put the ſine of the angle ♎ $= p$, its co-ſine $= q$, the arch ♎S $= z$, its Fig. 9.

its ſine $= x$, its coſine $= y$, and the length of the ſemi-periphery ♎D♈ $= e$: then, *p. ſpherics*, it will be, co-ſin. AS × co-ſin. ♎A ($= y \times Rad.$) $= y$; and ſin. AS ($= \frac{px}{Rad.}$) $= px$: whence, by multiplying theſe two equations together, we have ſin. AS × co-ſin. AS × co-ſin. ♎A ($=$ mn × co-ſin. ♎A) $= pxy$: and ſo, by ſubſtituting this value for its equal, our expreſſion for the nutation, during the time t' (given by the laſt Problem) will here be reduced to $360° \times \frac{3tt'}{2TT} \times kpxy$.

But the time wherein the ſun's longitude ♎S is augmented by the particle $\dot{z}$ will be $T \times \frac{\dot{z}}{2e}$; which being wrote in the room of t', we thence have $360° \times \frac{3t}{4T} \times \frac{kp}{e} \times xy\dot{z}$: and *this*, by ſubſtituting $\sqrt{1 - xx}$, and $\frac{\dot{x}}{\sqrt{1-xx}}$, inſtead of their equals y and $\dot{z}$, will be farther transformed to $360° \times \frac{3t}{4T} \times \frac{kp}{e} \times x\dot{x}$. Whoſe fluent, $360° \times \frac{3t}{4T} \times \frac{kpxx}{2e}$, is conſequently the true quantity of the nutation that was to be determined.

Again, with regard to the preceſſion of the equinox, the increaſe thereof *(by the Corol. to the precedent)* being in proportion to the correſponding decrement of the inclination of the equator to the ecliptic, as the tangent of ♎A to the ſine of ♎, or, in ſpecies, as $\frac{qx}{\sqrt{1-xx}}$ to p, it therefore appears (by multiplying the fluxion of the nutation by $\frac{qx}{p\sqrt{1-xx}}$) that $360° \times \frac{3t}{4T} \times \frac{kq}{e} \times \frac{x^2\dot{x}}{\sqrt{1-xx}}$ will be the fluxion of the quantity under conſideration; whoſe fluent, which is $360° \times \frac{3t}{4T} \times \frac{kq}{e} \times \frac{z - x\sqrt{1-xx}}{2}$, is therefore the preceſſion itſelf. *Q. E. I.*

COROL.

COROLLARY I.

When the ſun arrives at the ſolſtitial point D, the value of x being $= 1$, and that of $z = \frac{1}{2}e$, the quantity of the preceſſion becomes barely equal to $360° \times \frac{3t}{4T} \times \frac{kq}{4}$; whoſe quadruple, $360° \times \frac{3t}{4T} \times kq$, will be the whole of the annual preceſſion, depending on the ſun; which, in numbers (by making $t = 1$, $T = 366\frac{1}{4}$, $q = .91723 =$ co-ſine of $23° 28\frac{1}{2}'$, $a = 231$, $b = 230$, and k $(= \frac{aa - bb}{aa}) = \frac{2}{230\frac{1}{2}})$ comes out $21'' \, 7'''$. But it will appear from what follows hereafter, that this quantity, derived on the hypotheſis of an uniform denſity of all the parts of the earth, ought to be reduced to about $14\frac{1}{2}''$, to agree with obſervations.

COROLLARY II.

Since the preceſſion during $\frac{1}{4}$th of the annual revolution is found to be $360° \times \frac{3t}{4T} \times \frac{kq}{4}$, we have as $\frac{1}{2}e : z :: 360° \times \frac{3t}{4T} \times \frac{kq}{4} : (360° \times \frac{3t}{4T} \times \frac{kq}{2e} \times z)$ the mean preceſſion during the time of deſcribing the arch: which being taken from the true preceſſion, $360° \times \frac{3t}{4T} \times \frac{kq}{2e} \times \overline{z - x\sqrt{1 - xx}}$, the remainder, $360° \times \frac{3t}{4T} \times \frac{kq}{2e} \times - x\sqrt{1 - xx}$, will conſequently be the *equation* of the preceſſion; which therefore is to the mean preceſſion, as $- x\sqrt{1 - xx} : z$, or as $- 2x\sqrt{1 - xx} : 2z$; that is, as $-$ ſin. $2z : 2z$. But the mean preceſſion, in the time of deſcribing the arch z, is $\overline{21'' \, 7'''} \times \frac{z}{2e}$ (or rather $14\frac{1}{2}'' \times \frac{z}{2e}$) *by Corol. I.* Therefore the equation correſponding will be $- \frac{21'' \, 7'''}{4e} \times$ ſin. $2z = - \overline{1'' \, 42'''} \times$ ſin. $2z$, when the denſity is taken as uniform; but when taken to correſpond with the obſervation, it will be $- \frac{14\frac{1}{2}''}{4e} \times$ ſine $2z = - 1'' 10'' \times$ ſine $2z$. Hence it appears, that the greateſt equation of the preceſſion (when the ſun is in the mid-way between the equinox and ſolſtice) is 1″ 10‴; and that the general equation (which is ſubtractive in the firſt and third quadrants of the ecliptic) will be in proportion to the ſaid greateſt equation, as the ſine of twice the ſun's diſtance from the equinoctial point is to the radius.

COROLLARY III.

Furthermore, becauſe the quantity of the nutation is, univerſally, equal to $360° \times \frac{3t}{4T} \times \frac{kpx^2}{2e}$, it will therefore be the greateſt poſſible, when the ſun is in the ſolſtice and x is the greateſt poſſible: after which it will decreaſe, according to the ſame law whereby it before increaſed; 'till, on the ſun's arrival at the other equinoctial point, it intirely vaniſhes, and the inclination is thereby reſtored to its firſt quantity. It is alſo evident that the quantity of the nutation will, in all circumſtances, be in proportion to the preceſſion, during $\frac{1}{4}$th of the ſun's revolution, as $\frac{px^2}{e}$ to $\frac{q}{2}$, or as $\frac{px^2}{q}$ to $\frac{e}{2}$, that is, as the product under the ſquare of the ſine of the ſun's longitude and the tangent of the inclination of the two planes of the equator and ecliptic, is to the length of an arch of 90 degrees. According to which proportion (taking the ſaid preceſſion $= \frac{1}{4}$th of $14\frac{1}{2}''$) the greateſt nutation comes out one *ſecond*, very near; the inclination of the two planes decreaſing from the time of the ſun's leaving the equinoctial points, to his arrival at the ſolſtices, and that in the duplicate ratio of the ſine of his diſtance from the ſaid equinoctial points.

It may be obſerved that, in order to avoid trouble, the quantities p and q are taken as conſtant; the error, or difference thence ariſing ſcarcely amounting to $\frac{1}{1000000}$th part of the whole value.

SCHOLIUM.

Sir ISAAC NEWTON, in finding the preceſſion of the equinox, conſiders the protuberant matter about the earth's equator, as a ring of moons, revolving uniformly round the center of the earth in 24 hours; and by virtue of that aſſumption, from the motion of the lunar nodes, before determined, he infers the motion of the nodes of the ſaid ring, and from thence the preceſſion of the equinox. We have proceeded upon other principles, and by a very different method; and it may be worth while to remark here, that, as the preceſſion of the equinox is deducible from the motion of the nodes of a ſatellite, ſo, on the contrary, the motion of the nodes of a ſatellite may be very eaſily deduced, as a Corollary, from our general *formula*, for the preceſſion of the equinox.

Thus if the value of b be ſuppoſed indefinitely ſmall (ſo that the figure of the earth, or ſpheroid, may be conceived as flat as poſſible) we ſhall have $k\ (= \frac{aa - bb}{aa}) = 1$; and the expreſſion in Corol. I. will then become $360° \times \frac{3t}{4T} \times q$, exhibiting the motion of the node of a ring, or of a number of concentric rings, during the time (T) of one whole revolution of the body about the ſun *(vid. Corol. I. to Lem. II.)* But it will alſo appear, from the articles here referred to, that the place of a ſatellite, moving in a circular orbit, will always be found in a ring or plane revolving in the manner there ſpecified. Hence $360° \times \frac{3t}{4T} \times q$ will likewiſe expreſs the motion of the node of a ſingle moon, or ſatellite, in the time (T) of one whole revolution of the primary planet: which value, when the inclination to the ecliptic is but ſmall, will be equal to $360° \times \frac{3t}{4T}$, nearly. *Hence the mean motion of the node of a ſatellite, in a circular orbit, is in proportion to the mean motion of the primary planet about the ſun, as* $\frac{3}{4}$*ths of the periodic time of the ſatellite, is to the whole periodic time of the primary planet.* It follows moreover, that, let a planet have ever ſo many ſatellites, *the mean motions of the nodes of them all will be in proportion, directly as the times of revolution of the ſatellites themſelves; and, conſequently, the periodic times of the nodes, inverſly, as the periodic times of the reſpective ſatellites.*

The proportions uſed by Sir ISAAC NEWTON, in inferring the preceſſion of the equinox from the motion of the lunar node, agree exactly with thoſe above determined; it may, therefore, ſeem the more ſtrange that there ſhould be ſo wide a difference between the concluſion derived, in Corol. I. of the laſt Problem, and that brought out by that celebrated Author; who makes the quantity of the annual preceſſion, depending on the ſun, to be no more than $9'' \, 7''' \, 20^{iv}$: which is not the half of what it is here found to be. But to give the Reader what ſatisfaction I can in this particular, and to diſcover the error (if any ſuch ſhould have crept into my calculations) I ſhall now attempt the ſolution by a different method: in order to which it will be requiſite to premiſe the two following *Lemmas*.

LEMMA IV.

If every particle in the circumference AIaL *of a given circle tends to turn the circle about one of its diameters* IL, *as an axis, by a force proportional to the ſquare of the diſtance therefrom; it is propoſed to find the whole force of all the particles, to turn the circle about that diameter.*

If, from any point E in the circumference, there be drawn EH perpendicular to the given diameter IL, the force of a Fig. 10. particle at E will, *by hypotheſis*, be defined by EH^2; which quantity, drawn into (Em) the fluxion of the arch IE, will therefore be the fluxion of the quantity to be determined. But Em, becauſe of the ſimilar triangles Emn, EOH, will be equal to $\frac{OE \times En}{EH}$; and therefore the fluxion ſought $= OA \times EH \times En$. But $EH \times En$ is the fluxion of the area IHE: whence it is evident, that the force of all the particles in the whole circumference, will be truly defined by OA × *area of the circle, or* OA × OA × *the ſemi-circumference, or* $OA^2 \times \frac{1}{2}$ *the number of particles. Q. E. I.*

COROLLARY.

Since the force of a particle at A is expreſſed by OA^2, it follows that the force of all the particles in the whole circumference will be equal to half the force of an equal number of particles acting at the diſtance of the higheſt point A.

LEMMA

LEMMA V.

To determine the momentum of rotation of a given ſpheroid APaOp, *revolving uniformly about its axis* Pp, *with a given angular celerity.*

Let ENF be an ordinate to the generating ellipſis AP*ap*, parallel to the axis of rotation P*p*: make AO (perpendicular to P*p*) $= a$; OP ($=$ O*p*) $= b$; ON $= x$; EN $= y$; and let p denote the ſemi-periphery of the circle whoſe radius is unity.

Then it will be, as $1 : 2p :: x : 2px$, the periphery of the circle generated by the point N. Therefore $2px \times 2y$ will be the meaſure of the ſurface generated by the ordinate EF, in the revolution of the ellipſis about its axis P*p*: which, drawn into the ſquare of the diſtance ON, gives $4pyx^3$ for the momentum of rotation of all the particles in the ſaid ſurface: ſo that the fluent of $-4pyx^3\dot{x}$ will be the true meaſure of the force to be determined. Fig. 11.

Now, by the property of the ellipſis, we have $x^2 = \frac{aa}{bb} \times \overline{bb - yy}$; and conſequently $x\dot{x} = \frac{-aay\dot{y}}{bb}$: whence, by ſubſtituting $\frac{a^4}{b^4} \times \overline{b^2y\dot{y} - y^3\dot{y}}$ in the room of its equal $-x^3\dot{x}$, our fluxion is transformed to $\frac{4pa^4}{b^4} \times \overline{b^2y^2\dot{y} - y^4\dot{y}}$: whoſe fluent, when $y = b$, is found equal to $\frac{8pa^4b}{15}$.

COROLLARY.

Since $\frac{4pa^2b}{3}$ is known to expreſs the maſs or content of the ſpheroid, the momentum of rotation of any ſpheroid about its axis appears, therefore, to be juſt the ſame as would ariſe, if $\frac{2}{5}$ths of the whole maſs was to revolve at the diſtance of the higheſt point (A) from the axis of motion.

PROBLEM V.

To determine the alteration of the poſition of the terreſtrial equator, ariſing from the action of the ſun on the whole maſs of the earth, during an inſtant of time.

Fig. 12. Let OAP*ap* be the earth, under the form of an oblate ſpheroid; let AI*a*L be the plane of its equator, and HICL a plane paſſing thro' S the center of the ſun, making right-angles with the plane of the meridian HAPC*p* and with the plane GG. It is found, *in Prob. I.* that the force whereby a particle, at any point E in the equator AEL*a*I, tends from the plane GG, is in proportion to that reſpecting the higheſt point A, as the diſtance EF to the diſtance AK, or as ED to AO (ſuppoſing EF parallel to AK, and ED to AO); whence it is evident that the force on the particle at E, in a direction perpendicular to the plane of the equator, muſt be to the force on a particle at A, in the like direction, in the very ſame ratio of ED to AO, that is, in the ratio of the coſine of the arch AE to the radius. But this, *by Corol. I. Lem. II.* appears to be the law of the forces under which a ring of particles AEL*a*I, detached from the earth, may continue *in equilibrio*, in the ſame plane, under a twofold motion about the center O, and about the line A*a* as an axis.

Imagine now this ring to be exceeding denſe, ſo that its *momentum of rotation* about its center O, may be equal to that of the earth itſelf, or ſo that the two bodies may equally endeavour to perſevere in the ſame ſtate and direction of motion, in oppoſition to any new force impreſſed. Then it is evident, that, were the forces whereby the two bodies tend to turn about the line LI, through the ſun's attraction, to be *alſo* equal, the *ſame* effect, or alteration of motion, would be produced in both; and conſequently, that the effects produced, when the forces applied are unequal, will be in proportion directly as the forces. Now the force whereby a particle at A is urged from the plane GG, is found to be $\beta\times\frac{3tt}{TT}\times\frac{AK}{AO}$ *(by Prop. I. Corol. I.)*; which, in a direction perpendicular to the plane of the equator or ring, will be $\beta\times\frac{3tt}{TT}\times\frac{AK}{AO}\times\frac{OK}{AO}=\beta\times\frac{3tt}{TT}\times mn$.

Therefore, the force acting on a particle at E, in a like direction, being expreſſed by $\beta\times\frac{3tt}{TT}\times mn\times\frac{ED}{AO}$, the effect thereof to turn the ring about the line IL will be expreſſed by $\beta\times\frac{3tt}{TT}\times mn\times\frac{ED^2}{AO}$; which being as the ſquare of the diſtance ED, it follows *(from the Corol. to Lem. IV.)* that, if *M* be taken to denote the maſs of the ring, the whole force by which the ring tends to move about the line LI, as an axe, through the action of the ſun on all the particles, will be truly defined by $\beta\times\frac{3tt}{TT}\times mna\times\frac{1}{2}M$. Again, becauſe $\beta\times\frac{3tt}{TT}\times mn$ is the force acting on a particle at A, in a direction perpendicular to the plane of the ring, it is evident, *from Corol. I. to Lem. II.* that the ring will, in conſequence of that force, have a motion about the line A*a* as an axe; whoſe celerity will be to the celerity of the other motion about the center, in the proportion of *r* to 1, or of $\frac{3tt}{2TT}\times mn$ to 1; becauſe, $\beta\times\frac{3tt}{TT}\times mn$ being $=\beta\times 2r$, *r* will here be $=\frac{3tt}{2TT}\times mn$. Therefore, if *N* be aſſumed to denote the ſun's force to turn the earth about its center, we ſhall *(from the above obſervation)* have, $\beta\times\frac{3tt}{TT}\times mna\times\frac{1}{2}M$ (the force of the ring) : N :: $\frac{3tt}{2TT}\times mn$ (the motion of the ring) : $\frac{N}{\beta aM}$ = the required motion of the earth itſelf, about the ſame line A*a*.

But it appears, *by the Corol. to Lem. V.* that the maſs (M) of the ring (to have an equal momentum) muſt be juſt $\frac{2}{5}$ths of the maſs (S) of the earth: theref. our laſt expreſſion is equal to $\frac{N}{\beta a\times\frac{2}{5}S}$; which, by ſubſtituting $\beta\times\frac{3tt}{TT}\times\frac{mn\times\overline{aa-bb}}{5a}\times S$, in the room of its equal N, at laſt becomes $\frac{3tt}{2TT}\times\frac{mn\times\overline{aa-bb}}{aa}$; *being exactly the ſame as was before found in Prop. II.* The aſcertaining of which is the only real difficulty in the ſubject; ſince, that being once known, every thing that follows after is *purely* mathematical; nothing more being required than to take the ſum, or fluent of thoſe inſtantaneous alterations, in order to have the whole alteration, for any finite time propoſed; as is actually done in Prop. IV. which therefore it will be needleſs to repeat.

SCHO-

SCHOLIUM.

Fig. 12. From this laſt method it will not be very difficult to determine what the reſult ought to be, when the denſity, inſtead of being every-where the ſame, is ſuppoſed to increaſe or decreaſe from the ſurface to the center, according to a given law. For, let *N*, as above, be taken to denote the force whereby the earth tends to turn about its center, by the action of the ſun (the determination of which will be given by-and-by); and let *M*, alſo as before, expreſs the quantity of matter in an exceeding denſe ring, at the equator, having the ſame time of revolution, and *momentum of rotation*, with the earth itſelf; then it will appear, from the laſt Problem, that, let the figure and denſity of the earth be what they will, the celerity of the angular motion about the aforeſaid line A*a*, will be to that about the axis, in the ratio of $\frac{N}{\beta a M}$ to 1. Now, if the momentum of the earth about its axis (which I ſhall denote by *R*) be computed (by taking the ſum of the products of all the particles by the ſquares of their, reſpective, diſtances from the axis) the value of *M* will be known; becauſe the momentum of the ring (by the ſame rule) being $= M \times a^2$, we have $M \times a^2 = R$; and conſequently $\frac{N}{\beta a M} = \frac{aN}{\beta R}$; ſo that, *the general proportion between the celerities of the two motions, about the line* A*a*, *and the axe* P*p*, *will be that of* $\frac{aN}{\beta R}$ *to unity.*

But now, in order to find theſe values of *N* and *R*, according to any hypotheſis of denſity, we muſt look back to the third and fifth *Lemmas*; from the latter of which it appears, that the value of the momentum (*R*), when the denſity is uniform, will be truly defined by $\frac{8pa^4b}{15}$; *a* being the ſemi-equatoreal diameter, and *b* the ſemi-axis of the ſpheroid, and *p* the meaſure of the periphery of the circle whoſe radius is unity.

Now let us ſuppoſe the ſpheroid, inſtead of being everywhere of the ſame denſity, to be compoſed of *elliptical ſtrata*, whoſe denſities vary according to any given law of the diſtances of ſuch *ſtrata* from the center of the ſpheroid.

Then, putting $z =$ the ſemi-equatoreal diameter of any ſuch *ſtratum*, and ſuppoſing the correſponding ſemi-axis to be in proportion thereto, as w to 1 (which proportion may be aſſumed, either, as conſtant or variable), we ſhall, by writing z in the room of a, and $\overline{wz}$ in the room of b, have $\frac{8p}{15} \times wz^5$; whoſe fluxion, or indefinitely ſmall increment, will, therefore be the momentum of the *ſtratum* or ſhell, whereof the ſemi-equatoreal diameter is z, and thickneſs (at that diameter) $\dot{z}$; on the former ſuppoſition, that the ſpheroid is every-where of the ſame denſity. But in the preſent caſe this momentum muſt be drawn into the quantity, which, according to hypotheſis, expreſſeth the meaſure of the denſity anſwering to the value of z; which quantity we will repreſent by D; then will the product $\frac{8pD}{15} \times$ flux. wx^5 be the general fluxion of the momentum, when the denſity is variable: and therefore the fluent of this laſt expreſſion, when $z = a$, and $w = \frac{b}{a}$, will be the true value of the required momentum (R) in the preſent caſe.

After the ſame manner, the correſponding value of N *(from Lem. III.)* may be determined: for, retaining the above notation and ſuppoſitions, it is evident *(from thence)* that the ſaid value of N (which is there expreſſed by $\frac{\gamma}{a} \times \frac{mn}{5} \times \overline{a^2 - b^2} \times S$, or $\frac{\gamma}{a} \times \frac{mn}{5} \times \overline{a^2 - b^2} \times \frac{4pa^2b}{3}$) will here, by ſubſtitution, become $\frac{\gamma}{a} \times \frac{4mnp}{15} \times \overline{wz^5} - \overline{w^3z^5}$; and conſequently that $\frac{\gamma}{a} \times \frac{4mnp}{15} \times D \times$ flux. $\overline{wz^5 - w^3z^5}$ will be the required fluxion of the value of N: where $\frac{\gamma}{a}$ is a conſtant quantity (by hypotheſis), the force being proportional to the diſtance, and the meaſure thereof (γ) at the given diſtance a, equal to $\beta \times \frac{3tt}{TT}$, as has been before ſhewn, in Prop. II.

Now from the whole of what is above laid down, it will appear evident, that the angular celerity of the motion about the line Aa (ſuppoſing that about the axis to be denoted by unity) will be truly defined by $\frac{3mntt}{2TT} \times \frac{\text{fluent of } D \times \text{flux. } wz^5 - w^3z^5}{\text{fluent of } D \times \text{flux. } wz^5}$; which, therefore, will be known, when the relation of z, w, and D is aſſigned.

If w be ſuppoſed conſtant, or, which is the ſame, if all the *ſtrata* are conceived to be ſimilar to one another, then our expreſſion will become $\frac{3mntt}{2TT} \times \frac{\overline{w - w^3} \times \text{fluent of } D \times \text{flux. } z^5}{w \times \text{fluent of } D \times \text{flux. } z^5} = \frac{3mntt}{2TT} \times \frac{w - w^3}{w} = \frac{3mntt}{2TT} \times \overline{1 - w^2} = \frac{3tt}{2TT} \times \frac{mn \times \overline{aa - bb}}{aa}$ (becauſe $w = \frac{b}{a}$): which concluſion appears to be the very ſame with that found when the denſity was ſuppoſed uniform. From whence it is evident that an increaſe or decreaſe of denſity, in going towards the center, makes no ſort of difference *here*; provided the ſurfaces of the ſeveral *ſtrata* are all ſimilar to one another and to the ſurface of the earth. If indeed the *ſtrata* are *diſſimilar*, the caſe will be otherwiſe; as will be ſeen by the following example: which ought not be looked upon as a matter of mere ſpeculation; ſince it will appear, in the ſequel, that the preceſſion of the equinox cannot be accounted for, ſo as to agree with the *phænomenon*, upon the ſuppoſition of an uniform denſity of all the parts of the earth; the reſult, this way, coming out about $\frac{1}{3}$d part greater than the real quantity, determined by obſervation.

Let then, as before, the greateſt ſemi-diameter of any ſtratum be denoted by z, and let the leaſt ſemi-diameter (lying in the axe of the earth) be in proportion thereto as $1 - \lambda z^{\phi}$ to unity; alſo let the denſity be ſuppoſed to increaſe, in approaching the center of the earth, in the ratio of $\pi - \overline{\pi - 1} \times \frac{z^v}{a^v}$; ſo as to vary according to ſome power (z^v) of the diſtance, and that the meaſure thereof at the center, may be to that at the ſurface, in any given ratio of π to 1. Then, by taking λ, ϕ, and v, as conſtant quantities, and writing $1 - \lambda z^{\phi}$, and $\pi - \overline{\pi - 1} \times \frac{z^v}{a^v}$, inſtead of their equals w and D, we ſhall here have

$D \times \text{flux.}\,\overline{wz^5 - w^3x^5} = \overline{\pi - \overline{\pi - 1} \times \frac{z^v}{a^v}} \times \overline{\varphi + 5} \times 2\lambda z^{\varphi+4}\dot{z}$, nearly (becauſe, to render the calculus leſs laborious, the terms involving λ^2 and λ^3 may be here neglected as inconſiderable); the fluent of which expreſſion, when $z = a$, will be found $= \pi \times 2\lambda a^{\varphi+5} - \frac{\overline{\pi - 1} \times \overline{\varphi + 5} \times 2\lambda a^{\varphi+5}}{\varphi + v + 5} = \frac{v\pi + \varphi + 5}{v + \varphi + 5} \times 2\lambda a^{\varphi+5}$.

Moreover we have, in this caſe, $D \times$ fluxion of $wz^5 = \overline{\pi - \overline{\pi - 1} \times \frac{z^v}{a^v}} \times \overline{5z^4\dot{z} - \overline{5 + \varphi} \times \lambda z^{\varphi+4}\dot{z}} = \overline{\pi - \overline{\pi - 1} \times \frac{z^v}{a^v}} \times 5z^4\dot{z}$, nearly (becauſe $\overline{5 + \varphi} \times \lambda z^{\varphi+4}$, as the earth is nearly ſpherical, is inconſiderable in reſpect of $5z^4\dot{z}$); whereof the fluent, when $z = a$, will be had $= \pi a^5 - \frac{\overline{\pi - 1} \times 5a^5}{v + 5} = \frac{v\pi + 5}{v + 5} \times a^5$. Now let theſe two values be ſubſtituted in the general expreſſion $\frac{3mntt}{2TT} \times \frac{\text{fluent of } D \times \text{flux.}\,\overline{wz^5 - w^3z^5}}{\text{fluent of } D \times \text{flux.}\,wz^5}$; by which means it becomes $\frac{3mntt}{2TT} \times \frac{\overline{v\pi + \varphi + 5} \times \overline{v + 5}}{\overline{v + \varphi + 5} \times \overline{v\pi + 5}} \times 2\lambda a^{\varphi}$. But, when $z = a$, $w \left(= \frac{b}{a}\right)$ will be $= 1 - \lambda a^{\varphi}$, and $w^2 = 1 - 2\lambda a^{\varphi}$, nearly; whence we have $2\lambda a^{\varphi} = 1 - w^2 = 1 - \frac{bb}{aa} = \frac{aa - bb}{aa}$; and ſo, by ſubſtitution, our laſt *formula* becomes $\frac{3mntt}{2TT} \times \frac{\overline{v\pi + \varphi + 5} \times \overline{v + 5}}{\overline{v + \varphi + 5} \times \overline{v\pi + 5}} \times \frac{aa - bb}{aa}$. Whence it appears that the required motion, in this caſe, is to the motion, when the denſity is ſuppoſed uniform, in the proportion of $\frac{\overline{v\pi + \varphi + 5} \times \overline{v + 5}}{\overline{v + \varphi + 5} \times \overline{v\pi + 5}}$ to *unity*.—From this proportion a great number of Corollaries may be drawn; but theſe will be, more properly, conſidered hereafter.

PROBLEM VI.

To find the quantity of the preceſſion of the equinox, and alſo that of the nutation of the earth's axis, cauſed by the moon, during the time of half a revolution in her orbit.

Fig. 13. Let *f*FN*e*E be the orbit of the moon (on the ſurface of the ſphere) interſecting the ecliptic ♎♑♈ in N; and ſuppoſe F♎DE♈ to be the poſition of the equator, on the moon's paſſing it at F, and *fb*D*ea* the poſition thereof when ſhe repaſſeth it again at *e:* let moreover the quantity of the annual preceſſion ariſing from the ſun (given by Prob. IV.) be denoted by *A*; and let the ratio of the denſities of the moon and ſun be expreſſed by that of *m* to *unity:* then, taking τ to repreſent the given time in which the moon is moving from F to *e*, the mean quantity of the preceſſion, ariſing from the ſun, in that time, will be $\frac{\tau}{T}\times A$: and therefore, ſince the perturbating forces of the ſun and moon are as the denſities *(by Prob. I. Corol. II.)* it is evident that the preceſſion (E*e*) cauſed by the moon, in the ſame time, with reſpect to the plane of her own orbit, would be truly expreſſed by $m\times\frac{\tau}{T}\times A$, were the orbit's inclination to the equator to be always the ſame as *that* of the ecliptic to the equator: but, ſince the magnitude, as well as the poſition, of the angle E varies, with the place of the node, the ſaid quantity $m\times\frac{\tau}{T}\times A$ muſt therefore be diminiſhed in the ratio of the co-ſine of E to the co-ſine of ♈ (as appears by the ſaid Prob. IV.) and then we ſhall get $\frac{m\tau A}{T}\times\frac{\text{co-ſin. E}}{\text{co-ſin. ♈}}$, for the true value of the preceſſion E*e*, cauſed by the moon, with reſpect to her own orbit.

But now, in order to refer this to the ecliptic, it will be requiſite to obſerve firſt of all, that, as the inclination of the earth's axis, at the end of every half revolution, on the return of the ſun or moon again into the plane of the equator, is reſtored to its former quantity *(by Corol. III. to Prob. IV.)* it follows, ſeeing the angles E, *e*, F, *f* are thus equal, that the triangles DE*e* and D*f*F will alſo be equal and alike, in all reſpects; and ſo, DE + D*e* being = DE + DF = a ſemicircle, both DE and D*e* may be taken as quadrantal, or arcs of 90° each: whence, if ♑R, the meaſure of the angle ♈, be ſuppoſed to meet *a*ED in *r*, it will be, as *ſin.* ED *(radius)* : *ſin. e* (E) : : E*e* $\left(\frac{m\tau A}{T} \times \frac{\text{co-ſin. E}}{\text{co-ſin. ♈}}\right)$: ED*e* $= \frac{m\tau A}{T} \times \frac{\text{ſin. E} \times \text{co-ſ. E}}{\text{co-ſ. ♈} \times \text{rad.}}$;

alſo, as *ſin. a* (♈) : *ſin.* ♈D (co-ſin. ♈E) : : ED*e* : ♈*a* = $\frac{m\tau A}{T} \times \frac{\text{ſin. E} \times \text{co-ſin. E} \times \text{co-ſin. ♈E}}{\text{ſin. ♈} \times \text{co-ſin. ♈} \times \text{rad.}}$, the required quantity of the preceſſion:

And, as *ſin. r (radius)* : *ſin.* DR (♈E) : : ED*e* (RD*r*) : R*r* $= \frac{m\tau A}{T} \times \frac{\text{ſin. E} \times \text{co-ſin. E} \times \text{ſin. ♈E}}{\text{co-ſin. ♈} \times \overline{\text{rad.}}|^{2}}$, the correſponding quantity of the nutation, or the decreaſe of the inclination of the equator to the ecliptic. *Q. E. I.*

COROLLARY.

It is evident from hence that the quantity of the nutation is to that of the preceſſion, as $\frac{\text{ſin. ♈E}}{\text{rad.}}$ to $\frac{\text{co-ſin. ♈E}}{\text{ſin. ♈}}$, or as ſin. ♈ to $\frac{\text{rad.} \times \text{co-ſin. ♈E}}{\text{ſin. ♈E}}$; that is, as the ſine of ♈ to the co-tangent of ♈E. It appears moreover (becauſe *ſin.* E : *ſin.* ♈N : : *ſin.* N : *ſin.* ♈E, *p. ſpherics)* that the former of theſe quantities is alſo truly explicable by $\frac{m\tau A}{T} \times \frac{\text{ſin. N} \times \text{ſin. ♈N} \times \text{co-ſin. E}}{\text{co-ſin. ♈} \times \overline{\text{rad.}}|^{2}}$: which expreſſion will be of uſe in the following Problem.

PROBLEM VII.

To determine the preceſſion of the equinox, and the quantity of the nutation of the earth's axis, cauſed by the moon, during the time of half a revolution of the node of the moon's orbit.

Things being ſuppoſed as in the preceding Problem, let the diſtance ♈N of the node from the equinoctial point be denoted by *z*, its ſine by *x*, and its co-ſine by *y*; let alſo the ſine of the angle N♈E be put = *a*, its coſine = *b*, the ſine of Fig. 14.

N $= c$, its co-ſine $= d$, the ſemi-periphery ♈ ♑ ♎ $= e$, and the time of half a revolution of the node $= R$.

If ♈Q be ſuppoſed perpendicular to NE, it will be, *p. ſpherics*, as co-ſin. N♈ (y) : radius (1) : : co-tang. N $\left(\frac{d}{c}\right)$: tang. N♈Q $= \frac{d}{cy}$; let this be denoted by h, then the ſecant of the ſame angle will be $= \sqrt{1+hh}$, its ſine $= \frac{h}{\sqrt{1+hh}}$, and its co-ſine $= \frac{1}{\sqrt{1+hh}}$: whence, by the known rules for finding the ſine and co-ſine of the difference of two angles, the ſine of E♈Q will alſo be had $= \frac{hb - a}{\sqrt{1+hh}}$, and its coſine $= \frac{ha + bb}{\sqrt{1+hh}}$. Whence, again, *p. ſpherics*, we have, as ſin. N♈Q : ſin. E♈Q : : co-ſin. N : co-ſin. E $= \frac{\overline{hb - a} \times d}{h} = bd - \frac{ad}{h} = bd - acy$; alſo, as co-ſin. N♈Q : co-ſin. E♈Q : : co-tang. N♈ $\left(\frac{y}{x}\right)$: cotang. ♈E $= \overline{b + ta} \times \frac{y}{x} = \frac{ad + bcy}{cx}$; becauſe $h = \frac{d}{cy}$.

But, by the Corol. to the laſt Problem, the quantity of the nutation for the time τ, is expreſſed by $\frac{m\tau A}{T} \times \frac{\text{ſin. N} \times \text{ſin.} ♈\text{N} \times \text{co-ſ. E}}{\text{co-ſin. N}♈\text{E} \times \overline{\text{rad.}}|^2}$; which, in algebraic terms (by ſubſtituting the above values), will become $\frac{m\tau A}{T} \times \frac{cx \times \overline{bd - acy}}{b}$. But the time τ, during which the longitude (z) of the node is increaſed by $\dot{z}$, being to R the time of half a revolution of the node, as $\dot{z}$ to e, its value will therefore be expounded by $R \times \frac{\dot{z}}{e}$, or its equal $R \times \frac{\dot{x}}{e\sqrt{1 - xx}}$: and ſo by ſubſtituting this value, and writing $\sqrt{1 - xx}$ in the room of y, our laſt expreſſion will be reduced to $\frac{mAR}{T} \times \frac{c}{eb} \times \overline{\frac{bdx\dot{x}}{\sqrt{1-xx}} - acx\dot{x}}$; whoſe fluent $\frac{mAR}{T} \times \frac{c}{eb} \times \overline{bd - bd\sqrt{1-xx} - \frac{1}{2}acxx}$

(=

$(= \frac{m\text{AR}}{\text{T}} \times \frac{c}{be} \times \overline{bd \times \text{verſed ſine } z - \frac{1}{4}ac \times \text{verſed ſine } 2z})$ is conſequently the meaſure of the nutation, or the decreaſe of the inclination of the equator to the ecliptic, cauſed by the moon, from the time of the node's coinciding with the equinoctial point ♈, to its arrival at the poſition N.

Again, with regard to the preceſſion of the equinox, the increaſe thereof being in proportion to the decrement of the inclination, as the co-tangent of ♈E to the ſine of N♈E *(by the Corollary to the precedent)* or, in ſpecies, as $\frac{ad + bcy}{cx}$ (or $\frac{ad + bc\sqrt{1 - xx}}{cx}$) to a, its fluxion will therefore be had by multiplying that of the nutation, given above, into $\frac{ad + bc\sqrt{1 - xx}}{acx}$; and ſo is found to be

$$\frac{m\text{AR}}{\text{T}} \times \frac{1}{abe} \times \overline{\frac{abd^2\dot{x}}{\sqrt{1 - xx}} + \overline{bb - aa} \times cd\dot{x} - abc^2\dot{x}\sqrt{1 - xx}}\,;$$

whoſe fluent, which is

$$\frac{m\text{AR}}{\text{T}} \times \frac{1}{abe} \times \overline{abd^2z + \overline{bb - aa} \times cdx - \tfrac{1}{2}abc^2z - \tfrac{1}{2}abc^2x\sqrt{1 - xx}}$$

$$(= \frac{m\text{AR}}{\text{T}} \times \frac{1}{abe} \times \overline{\overline{dd - \tfrac{1}{2}cc} \times abz + \overline{bb - aa} \times cd \times \text{ſ.}\,z - \tfrac{1}{4}abcc \times \text{ſ.}\,2z}),$$

muſt conſequently be the preceſſion itſelf.

But, at the end of half a revolution, when the node N arrives at the other equinoctial point ♎, both this, and the expreſſion for the nutation, will become much more ſimple, x being then $= 0$, and $z = e$; whence the nutation will be $= \frac{m\text{AR}}{\text{T}} \times \frac{c}{eb} \times 2bd = \frac{m\text{AR}}{\text{T}} \times \frac{2cd}{e}$; and the preceſſion equal to $\frac{m\text{AR}}{\text{T}} \times \overline{dd - \frac{1}{2}cc} = \frac{m\text{AR}}{\text{T}} \times \overline{1 - \frac{3}{2}cc}$ (becauſe $cc + dd = 1$). *Q. E. I.*

COROLLARY I.

It appears from hence, that the mean preceſſion of the equinox, ariſing from the action of the moon, is in proportion to what it would otherwiſe be, if the moon's orbit was to coincide with the ecliptic, as $1 - \frac{3}{2}cc$ to *unity:* whence the true value thereof is to that depending on the ſun, in a ratio compounded of the ratio of the denſity of the moon to the denſity of the ſun, and the aforeſaid ratio of $1-\frac{3}{2}cc$ (or 0,988) to *unity*.

COROLLARY II.

It appears likewiſe, that the whole quantity of the nutation, in half a revolution of the node, is to the correſponding quantity of the preceſſion, as $\frac{2cd}{e}$ to $dd-\frac{1}{2}cc$, or as *unity* to $\overline{\frac{d}{c}-\frac{\frac{1}{2}c}{d}}\times\frac{e}{2}$; that is, as the radius to the exceſs of the cotangent, above half the tangent of the orbit's inclination, drawn into (1.5708) the meaſure of half the periphery of the circle whoſe diameter is *unity*. This proportion, in numbers, ſuppoſing the mean inclination of the orbit to be 5° 8′, will be found to be as 10 to 174, very near.

COROLLARY III.

Moreover, ſeeing the preceſſion in half a revolution of the node is $\frac{m\mathrm{AR}}{\mathrm{T}}\times\overline{dd-\frac{1}{2}cc}$, we have, as $e:z::\frac{m\mathrm{AR}}{\mathrm{T}}\times\overline{dd-\frac{1}{2}cc}:\frac{m\mathrm{AR}}{\mathrm{T}}\times\frac{z}{e}\times\overline{dd-\frac{1}{2}cc}$, the quantity of the mean preceſſion during the time in which the node moves over the arch z, or ♈N. This being ſubtracted from the true preceſſion, found above, the remainder $\frac{m\mathrm{AR}}{\mathrm{T}}\times\frac{1}{abe}\times\overline{\overline{bb-aa}\times cdx-\frac{1}{2}abc^2x\sqrt{1-xx}}$ will conſequently be the *equation of the preceſſion*, or the exceſs of the true above the mean: which equation or exceſs, if we neglect the term $-\frac{1}{2}abc^2x\sqrt{1-xx}$ (whoſe value, by reaſon of the ſmallneſs of c^2, never amounts to $\frac{1}{4}$th of a *ſecond)* will evidently be at its greateſt value at the end of $\frac{1}{4}$th of a revolution, on the node's arrival at the ſolſtice; when it becomes $\frac{m\mathrm{AR}}{\mathrm{T}}\times\frac{1}{abe}\times\overline{bb-aa}\times cd$; and, is therefore, in proportion to $\frac{m\mathrm{AR}}{\mathrm{T}}\times\frac{2cd}{e}$, the whole, or greateſt quantity of the nutation, during half a revolution of the node, as $bb-aa:2ab$, or as $1 : \frac{2ab}{bb - aa}$, that is, as the radius to the tangent of double the inclination of the equator to the ecliptic.

COROLLARY IV.

Furthermore, ſince the value of c (the ſine of the orbit's inclination) is but ſmall, the laſt term of the general expreſſion for the nutation, as well as that for the exceſs of the true preceſſion above the mean, may be rejected, without producing any conſiderable error; whence the nutation is reduced to $\frac{mAR}{T} \times \frac{cd}{e} \times \overline{1 - \sqrt{1 - xx}}$, and the preceſſion to $\frac{mAR}{T} \times \frac{1}{abe} \times \overline{bb - aa} \times cdx$. Hence it appears that the decreaſe of the inclination, from the time of the node's leaving the equinoctial point ♈, will be as the verſed ſine $(1 - \sqrt{1 - xx})$ of the node's true longitude; and that the exceſs of the true preceſſion above the mean, will be always as the ſine (x) of the ſame longitude.

SCHOLIUM.

The quantity of the annual preceſſion of the equinox ariſing from the force of the ſun, is found in Prob. IV, to be $21'' \, 7'''$; upon the ſuppoſition of all the parts of the earth being homogenous, and in a ſtate of fluidity. If, therefore, this quantity be taken from ($50''$) the whole, *obſerved*, annual preceſſion, ariſing from the ſun and moon conjunctly, the remainder $28'' \, 53'''$ will conſequently be the mean annual preceſſion depending on the moon; which being increaſed in the ratio of 1000 to 988 (according to Prob. VII. Corol. I.) gives $29'' 14'''$, for the quantity of the preceſſion, if the orbit of the moon were to coincide with the plane of the ecliptic. Hence it will be *(by the ſame Corol.)* as $21'' 7'''$ is to $29'' 14'''$, ſo is the denſity of the ſun to the denſity of the moon, *according to this hypotheſis*. But it is evident from experience (whether we regard the proportion of the tides, or the accurate obſervations of Dr. BRADLEY) that the denſity of the moon in reſpect to *that* of the ſun, cannot be ſo ſmall as it is here aſſigned.

It is true, there is no way of knowing the *exact* ratio of the denſities of the two luminaries; ſince *theory*, for want of ſufficient *data*, fails us here. And as to the method, *by obſerving and comparing the ſpring and neap tides* * (whether we regard the quantities or times *of them)* it cannot be otherwiſe than very precarious; conſidering the many obſtacles and intervening cauſes by which *they* are perpetually, more or leſs, influenced and diſturbed. Upon the whole it therefore ſeems to me, that the beſt method to ſettle this point (as far as the nature of the ſubject will allow of) is from the *obſerved* quantity of the nutation itſelf; agreeable to what has been hinted on this head by that celebrated Aſtronomer, to whoſe accurate obſervations we owe this important diſcovery.

Let us, therefore, take g to denote the greateſt nutation of the earth's axis, *as given by obſervation*; and then, if f be taken to repreſent the mean annual preceſſion, *given in like manner*, it will appear *(by Prob. VII. Corol. II.)* that $\frac{174}{10} \times \frac{1}{9\frac{3}{10}} \times g$ is the part of the ſaid annual preceſſion depending on the moon; whence the remaining part, owing to the ſun, muſt neceſſarily be $f - \frac{174}{93} \times g \left(= \frac{31f - 58g}{31}\right)$. Therefore we have *(by Problem VII. Corol. I.)* as $\frac{58g}{31} \times \frac{1000}{988}$ to $\frac{31f - 58g}{31}$, or as $\frac{58g}{31f - 58g} \times \frac{1000}{988}$ to 1, ſo is the denſity of the moon to the denſity of the ſun; which, in numbers (making $g = 18''$), will come out as 2,09 to 1. But if the value of g be ſuppoſed only a ſecond or two greater or leſs than 18″, the reſult will be ſenſibly different, as may be ſeen in the annexed Table; wherein, beſides the ratio of the denſities, are alſo exhibited the mean quantities of the annual preceſſion, depending on the forces of the ſun and moon, reſpectively; together with the greateſt equation of the ſaid preceſſion, *as given by Problem VII. Corollary III.*

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* Sir ISAAC NEWTON, by this method, makes the proportion to be as $4\frac{1}{2}$ to 1; and M. DANIEL BERNOULLI, only as $2\frac{1}{2}$ to 1.

Greateſt

Greateſt nutation.	Ratio of the denſities of the ſun and moon.	Annual preceſſion cauſed by the ſun.		Mean annual preceſſion cauſed by the moon.		Greateſt equation of the preceſſion cauſed by the moon.	
Seconds		Seconds	Thirds	Seconds	Thirds	Seconds	Thirds
16	1 : 1,51	20	3	29	57	14	58
17	1 : 1,77	18	11	31	49	15	54
18	1 : 2,09	16	19	33	41	16	50
19	1 : 2,50	14	27	35	33	17	46
20	1 : 3,01	12	35	37	25	18	42

Were I to deliver my opinion which of the different caſes here put down anſwers beſt to the *phænomenon*, and the general law of gravitation, I ſhould, without heſitating, fix upon that preceding the laſt; which, upon the whole, will be found to agree better with Dr. BRADLEY's obſervations than any of the others: beſides, though the obſervations on the tides cannot be relied on to any great degree of exactneſs, yet, by them, it is ſufficiently evident, that the perturbating force of the moon cannot be to *that* of the ſun in a leſs proportion than of about $2\frac{1}{2}$ to 1.

From the greateſt nutation, and the greateſt equation of the preceſſion, *given above*, the quantity of the nutation and the equation of the preceſſion, correſponding to any given poſition of the lunar node, may be very eaſily determined: for, *firſt*, it will be, *by Corol. IV.*

As the radius is to the ſine of the node's diſtance from the neareſt equinoctial point, ſo is the greateſt equation of the preceſſion to the equation ſought.

Which muſt be added to the mean preceſſion when the node (*viz.* the aſcending one) is in any of the ſix ſouthern ſigns; but ſubtracted, when in any of the ſix northern ones.

Secondly, it will appear, *by the ſame Corollary*, that the decreaſe of the inclination of the equator to the ecliptic, from the time of the node's coinciding with the equinoctial point ♈, is proportional to the verſed ſine of the node's preſent diſtance from that point: whence it follows, that the ſaid inclination will be at its mean value when the node is in the ſolſtice; and conſequently, that the difference between the mean, and true values, will be as the difference between the verſed ſine of the node's preſent diſtance from ♈, and the verſed ſine of 90 degrees, that is, as the co-ſine of the node's diſtance from ♈. Therefore, to find the nutation at any given time, it will be,

As the radius is to the co-ſine of the node's diſtance from the neareſt equinoctial point, ſo is the greateſt nutation to the nutation ſought.

Which, to have the true obliquity of the equator to the ecliptic, muſt be added, when the node is in any of the ſix aſcending ſigns ♑, ♒, ♓, ♈, ♉, ♊; but, otherwiſe, ſubtracted.

The following Table, ſhewing by inſpection, as well the equation of the preceſſion, as that of the obliquity of the ecliptic, is computed from the proportions here laid down; upon ſuppoſition that the greateſt quantity of the nutation is 19 ſeconds.

The Equation of the Preceſſion of the Equinox.				
☽'s ☊	Sig. 0	Sig. I	Sig. II	Subtr.
from ♈	Sig. VI	Sig VII	Sig. VIII	Add
Deg.	Seconds	Seconds	Seconds	Deg.
0	0,0	8,8	15,3	30
5	1,5	10,1	16,1	25
10	3,0	11,4	16,7	20
15	4,5	12,5	17,2	15
20	6,0	13,6	17,5	10
25	7,4	14,5	17,7	5
30	8,8	15,3	17,7	0
Subtr.	Sig. V	Sig. IV	Sig. III	☽'s ☊
Add	Sig. XI	Sig. X	Sig. IX	from ♈

The Equation of the Obliquity of the Ecliptic.				
☽'s ☊	Sig. 0	Sig. I	Sig. II	Add
from ♈	Sig. VI	Sig. VII	Sig. VIII	Subtr.
Deg.	Seconds	Seconds	Seconds	Deg.
0	9,5	8,2	4,7	30
5	9,4	7,8	4,0	25
10	9,3	7,3	3,3	20
15	9,2	6,7	2,4	15
20	9,0	6,1	1,7	10
25	8,7	5,5	0,9	5
30	8,2	4,7	0,0	0
Subtr.	Sig. V	Sig. IV	Sig. III	☽'s ☊
Add	Sig. XI	Sig. X	Sig IX	from ♈

Fig. 15. To place what has been delivered above in another view, ſuppoſe PE to be equal to the mean diſtance of the pole of the equator from the pole E of the ecliptic; in which (produced) let there be taken PA and PB equal, each, to half the greateſt nutation; and about AB, as an axis, conceive an ellipſe ACBF to be deſcribed; whoſe other axe CF is to AB in the ratio of the co-ſine of 2EP to the co-ſine of EP (that is, in numbers, as 7444 to 10000, or as 3 to 4, nearly); then, if the point P repreſents the mean place of the pole of the equator, the true place will always be found in the circumference of the ſaid ellipſe. And if, on the diameter AB, a circle ADBG be alſo deſcribed, and the angle APS be made equal to the diſtance of the node from the equinoctial point ♈; then, I ſay, a perpendicular SR*p*, falling from the point S upon the diameter AB,

will interſect the circumference of the ellipſe in the point where the pole of the equator (p) is, at that time, poſited. For, firſt, it is clear, from what has been already remarked, that AE and BE will be the greateſt and leaſt diſtances of the two poles, as being equivalent to the reſpective inclinations of the two planes, the equator and the ecliptic: from whence and Corol. IV. it is manifeſt, that ER or Ep will be the true diſtance of the ſaid poles, when the verſed ſine of the node's diſtance (APS) from ♈ is AR. Moreover, *by conſtruction*, CP : AB :: $\frac{1}{2}$ the co-ſine 2EP : co-ſin. EP; that is, in ſpecies, CP : AB :: $\frac{bb-aa}{2}$: b. And, *p. ſpherics*, tang. PEC : tang. PC (:: PEC : PC, nearly) :: rad. (1) : a. Therefore, by compounding theſe two proportions, we have PEC : AB :: $\frac{bb-aa}{2}$: ab :: $bb-aa$: $2ab$: which proportion, for finding the angle PEC, is the very ſame with that determining the greateſt difference of the mean, and true longitudes, as given by *Corol. III.* Whence it eaſily follows, that the angle REp will expreſs the difference of the mean and true longitudes, at the given poſition of the node; ſince, as the radius : ſine APS (:: PD : RS :: PF : Rp) :: the angle PEC : the angle REp, as it ought to be, *by Corol. IV.* The ratio of CF to AB is here determined to a geometrical exactneſs, as no-ways depending, either, on the denſity of the moon, or on any other phyſical hypotheſis.

Having now laid down the general proportions for the nutation of the earth's axis, and the preceſſion of the equinox, I ſhall here ſubjoin the neceſſary rules for determining how much the declinations and right-aſcenſions of the ſtars are affected by thoſe inequalities.

1°. For the alteration of a ſtar's declination, and right-aſcenſion, ariſing from the nutation of the earth's axis; it will be

As the radius is to the ſine of the ſtar's right-aſcenſion, ſo is the nutation (or the given alteration of the equator's inclination to the ecliptic) to the alteration of the ſtar's declination, cauſed by the nutation;

And, as the co-tangent of the ſtar's declination is to the co-ſine of its right-aſcenſion, ſo is the nutation to the alteration of the ſtar's right-aſcenſion, correſponding.

2°. For the alteration of the ſtar's declination and right-aſcenſion, ariſing from the preceſſion of the equinox; it will be

As the co-ſecant of the obliquity of the ecliptic is to the co-ſine of the ſtar's right-aſcenſion, ſo is the preceſſion of the equinox (or the alteration of the ſtar's longitude) to the alteration of the ſtar's declination, cauſed by the preceſſion;

And as the co-ſine of the ſtar's declination is to the co-tangent of its angle of poſition, ſo is the alteration of declination, found by the laſt proportion, to the alteration of right-aſcenſion, anſwering thereto.

Any one, but little acquainted with the ſphere, will eaſily ſee *when* theſe equations are additive, and *when* ſubtractive: nor will it be at all difficult to comprehend the reaſons upon which they are founded; they being nothing more than ſo many particular caſes of the general relation ſubſiſting between the fluxions of the ſides and angles of a ſpherical triangle *. It will not, however, be improper to remark here, that, when the quantity of the preceſſion, in the ſecond of the preceding caſes, amounts to ſome *minutes*, it will be neceſſary, in order to have the concluſion ſufficiently exact, to make uſe of the mean right-aſcenſion, at the middle of the given interval; which, from the given right-aſcenſion at the beginning of the interval, may be eſtimated near enough for the purpoſe, in moſt caſes, without the trouble of a calculation: but in other caſes, and when the utmoſt exactneſs is required, it will be neceſſary to repeat the operation.

It may not be improper to obſerve likewiſe, that, beſides the equations depending on the poſition of the lunar nodes, computed above, there is a ſmall motion of nutation and preceſſion ariſing from the moon's declination; whereof the greateſt quantity is to the greateſt quantity of *that* depending on the ſun, in a ratio compounded of the ratio of the denſities of the two bodies, that of their periodic times, and that of the ſines of the inclinations of their reſpective orbits to the plane of the equator, nearly (as appears by Prob. IV. and VI.) Whence it is evident, that this part of the nutation, depending on the moon's declination, cannot, in any circumſtance, amount to more than about $\frac{1}{4}$th of a *ſecond*; a quantity too ſmall to merit attention in the practice of Aſtronomy.

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* See my *Doctrine of Fluxions*, Part II. Sect. I.

Remarks on ſome Particulars in the preceding Theory and Calculations; in order to explain and obviate certain difficulties and objections that may thence ariſe.

IT may be obſerved, in the firſt place, that we have, all along, conſidered the effects of the ſun and moon ſeparately; and, conſequently, have ſuppoſed them to be no-ways influenced or diſturbed by each other. This may ſeem too bold an aſſumption; eſpecially, as it is known that the tides, which are produced by the very ſame forces, depend upon, and are greatly varied by, the different poſitions of the two luminaries.

Fig. 16.

To remove this objection, let ♈SM♎ repreſent the plane of the earth's equator ♈O♎, its interſection with the plane of the ecliptic, ♈S the right-aſcenſion of the ſun, and ♈M the right-aſcenſion of the moon; and let the forces of the two bodies to turn the earth about its center, in thoſe poſitions, be repreſented by f and F, reſpectively.

Theſe forces may be conſidered as acting perpendicular to the plane of the equator in the points S and M, and will be equivalent to, and have the ſame effect with, one ſingle force, equal to them both, acting in their center of gravity N. But, by mechanics, the force $f+F$, acting at N, will (if the radius OP be drawn through N) be equivalent to another force, acting at P, expreſſed by $\overline{f+F}\times\frac{ON}{OP}$, or $\overline{f+F}\times\frac{NQ}{PR}$ (ſuppoſing NQ, PR, as alſo SB and MC, to be perpendicular to ♈O♎).

But the quantity of the preceſſion, during a given moment of time, is known to be as the force, and as the ſine of the right-aſcenſion, conjunctly *(by Prob. III.)*; from whence the two quantities ariſing from the ſun and moon, conſidered ſeparately, are expounded by $f\times SB$, and $F\times MC$, reſpectively. But, ſuppoſing both bodies to act together, or, which is the ſame, ſuppoſing one ſingle force, expreſſed by $\overline{f+F}\times\frac{NQ}{PR}$, to act at P, the quantity of the preceſſion will then (by the very ſame rule) be truly defined by $\overline{f+F} \times \frac{NQ}{PR} \times PR$, or its equal $\overline{f+F} \times NQ$; which quantity, by the property of the center of gravity, is known to be equal to $f \times SB + F \times MC$. Hence it is manifeſt that, whether the forces of the luminaries be joined together, or treated apart, the reſult will be the ſame.

The next difficulty, relates to the excentricity of the lunar orbit, and the inequality of the motion in that orbit; which may be thought ſufficient to occaſion a ſenſible deviation from rules founded on a ſuppoſition that pays no regard to them.

Fig. 17. In order to clear up this point alſo, imagine ADBE to be an ellipſe, in which the moon is ſuppoſed to revolve, about the center of the earth, placed in the lower focus F of the ellipſe: let AB be the tranſverſe axis of the ellipſe, perpendicular to which, through F, draw the ordinate IH; moreover let there be drawn any two other lines DE, *de*, through the focus F, to make a very ſmall (given) angle DF*d* with each other.

The perturbating force of the moon, at the diſtance DF, will *(by Prop. I. Corol. II.)* be, *inverſly*, as the cube of that diſtance; and the time of deſcribing the given angle DF*d* will, it is well known, be *directly* as the ſquare of the ſame diſtance. Therefore, by compoſition, the quantity of the moon's action, during the time of deſcribing this angle, will be in the ſimple ratio of the ſaid diſtance, *inverſly*. Hence it appears, that the ſum of the forces employed, during the times of deſcribing the oppoſite angles DF*d*, EF*e*, will be truly defined by $\frac{1}{FD} + \frac{1}{FE}$, or its equal $\frac{FE + FD}{FE \times FD}$.

Upon AB let fall the perpendiculars DN and EM; ſo ſhall $FE - FH : FI\,(FH) - FD :: FM : FN$ *(p. ellipſe)* $:: FE : FD$ *(p. ſim. triang.)*: conſequently $FE \times FD - \overline{FH \times FD} = FH \times FE - FD \times FE$, or $2FE \times FD = FH \times \overline{FE + FD}$: therefore, as it appears from hence that $\frac{FE + FD}{FE \times FD}$, the meaſure of the ſaid forces, is, every-where, equal to the conſtant quantity $\frac{2}{FH}$, it is evident that the excentricity of the orbit and the poſition of the apogee have no effect on the motion of the earth's axis.

An objection may, perhaps, ariſe, with regard to the addition of the forces employed by the moon in oppoſite parts of her orbit; which ſtep may be looked upon as arbitrary: but the reaſon upon which it is founded will be clear, by conſidering that the moon's inclination to the plane of the equator, in oppoſite points of her orbit, is always the ſame; and that, therefore, the very ſame effect in the alteration of the poſition of the equator will be produced, whether the whole force employed during the deſcription of the correſponding oppoſite angles, be equally, or unequally, divided, with reſpect to the ſaid angles; ſince the ſaid force acts with the ſame advantage, or under the ſame circumſtance of declination, in both caſes.

Another difficulty that may ariſe, is in relation to our having made the effect of the ſun's force to be about $\frac{1}{3}$ part leſs than the quantity reſulting from calculations founded on hydroſtatical principles and the hypotheſis of an uniform denſity of all the parts of the earth. But, that the *phænomenon* cannot be truly accounted for, upon this hypotheſis, appears from the concurrence of all experiments in general: for, whether we regard the menſuration of the degrees of the earth, the accurate obſervations of Dr. BRADLEY, or the proportions and times of the tides, the caſe is the ſame, and requires a much leſs effect from the action of the ſun than reſults from, or can conſiſt with, the ſaid hypotheſis.

But if the denſity of the earth, inſtead of being uniform, is ſuppoſed to increaſe from the ſurface to the center (as there is the greateſt reaſon to imagine it does), then the *phænomenon* may be eaſily made to quadrate with *the principles of gravitation*; and that according to innumerable ſuppoſitions, reſpecting the law whereby the denſity may be conceived to increaſe.

Thus, conformable to the hypotheſis laid down in the Scholium after Prob. V. the motion of the equinoctial points will be in proportion to the motion of the ſame points, when the denſity is ſuppoſed uniform, as $\frac{\overline{v\pi + \varphi + 5} \times \overline{v + 5}}{\overline{v + \varphi + 5} \times \overline{v\pi + 5}}$ to 1, that is, as $1 - \frac{v\phi\times\overline{\pi-1}}{\overline{v+\phi+5}\times\overline{v\pi+5}}$ is to 1: therefore, by making $1 - \frac{v\phi\times\overline{\pi-1}}{\overline{v+\phi+5}\times\overline{v\pi+5}} = 1 - \frac{1}{3}$ (agreeable to what has been above obſerved), we ſhall have $\frac{v\phi\times\overline{\pi-1}}{\overline{v+\phi+5}\times\overline{v\pi+5}} = \frac{1}{3}$: by means of which equation, the relation of π, ϕ, and v may be ſo aſſigned, as to give the true quantity of the preceſſion, and that innumerable ways. As one inſtance hereof, let us ſuppoſe $\pi = 2$, or that the denſity at the center is juſt double to *that* at the ſurface; and let the value of ϕ be ſuppoſed very great, or, which comes to the ſame, let the *ſtrata* in the lower parts of the earth, be ſuppoſed very nearly ſpherical, or orbicular: then our equation will become $\frac{v\phi}{\overline{v+\phi+5}\times\overline{2v+5}} = \frac{1}{3}$; which, becauſe ϕ is ſuppoſed very great, will be $\frac{v}{2v+5} = \frac{1}{3}$, nearly; whence v is given $= 5$: ſo that, according to this hypotheſis, the decreaſe of denſity, in going from the center of the earth to the ſurface, will be in the quintuplicate ratio of the diſtances from the center.

No One can imagine that we pretend here to aſcertain the ſtructure and denſity of the interior parts of the earth: all that is attempted, is to ſhew (which indeed is all that can be done) that the preceſſion of the equinox may be truly accounted for *upon the principles of gravitation*, though not in the hypotheſis of an uniform denſity of all the parts of the earth, unleſs by aſſuming the difference of the leaſt and greateſt diameters much ſmaller than it is found to be, either, from hydroſtatical principles, or by an actual menſuration of the degrees of the earth's meridian.

There remains yet another particular that I cannot avoid taking ſome further notice of; which is the wide difference to be found between our concluſion, in Prob. IV. Corol. I. and *that* brought out by Sir ISAAC NEWTON *(in Prop.* 39. *Book III. of his Principia)* from the very ſame *data*.

I am ſenſible that this is a delicate point to touch upon; but then I know likewiſe, that I might leave my Readers diſſatisfied, were not I to endeavour to point out the cauſes of the ſaid difference.—At firſt I had, myſelf, a ſtrong ſuſpicion that I had, ſomewhere, fallen into an error; which put me upon attempting the ſolution by different methods, as the moſt proper way to arrive at certainty, and to diſcover the miſtake, if any *ſuch* had crept into my calculations. Two of theſe methods I have given; the others ſeemed unneceſſary. The exact concurrence of them all, firſt made me think, that it was not impoſſible but there might be a fault in that Author's ſolution; and occaſioned my looking into his method with a more particular attention than I had before regarded it with.

What, at firſt, ſeemed moſt doubtful to me was his hypotheſis, *that the motion of the nodes of a ring would be the ſame, whether the ring were fluid, or whether it conſiſted of a hard rigid matter* *: this, I ſay, did not ſeem at all clear, at firſt; but upon recollecting the demonſtration of my ſecond Lemma (wherein this point is fully, though not directly, proved) I was ſoon convinced that the fault (if ſuch there was) muſt be owing to ſomething elſe.

In the next place, his third Lemma did not appear to me ſo well grounded as the two preceding ones. In this Lemma he determines, *that the motion of the whole earth about its axe, ariſing from the motion of all the particles, will be to the motion of a ring about the ſame axe, in a proportion compounded of the proportion of the matter in the earth to the matter in the ring, and of the number* 925725 *to the number* 1000000. This proportion is, indiſputably, true, in the ſenſe of the Author: but there is a difference between the quantity of motion, ſo conſidered, and the *momentum* whereby a body, revolving round an axis, endeavours to perſevere in its preſent ſtate of motion, in oppoſition to any new force impreſſed. Now it ſeems clear to me, that it is this laſt kind of *momentum* that ought to be regarded, in computing the alteration of the body's motion, in conſequence of any ſuch force. And here every particle is to be conſidered as acting by a lever terminating in the axis of motion: ſo that, to have the whole momentum ſought, the moving force of each particle muſt be multiplied into the length of the lever by which it ſuppoſed to act: whence the momentum of each particle will be proportional to the ſquare of the diſtance from the axis of motion; as it is known to be in finding the centers of percuſſion of bodies, which depend on the very ſame principles.

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* The celebrated mathematician M. D'ALEMBERT, who has with great ſubtlety expatiated on Sir ISAAC NEWTON's ſolution of this Problem, repreſents the above *hypotheſis*, as ill founded; and ſays, that, when the ring is in a fluid ſtate, the particles, or detached moons will not have their centers in one and the ſame plane *(il eſt certain que des lunes iſolées n'auroient pas toujours leurs centres placés dans un même plan)*. Now if, by this, we are to underſtand, that the deviation from a plane is ſomething ſenſible in compariſon of the nutation in queſtion, what is advanced is repugnant to what is demonſtrated in our ſecond Lemma. But if an exceeding ſmall deviation (depending on the ſecond term of a ſeries) be only intended (and ſuch it muſt be, if any thing at all), ſuch a ſuppoſition will make nothing againſt our Author's aſſumption; as, in phyſical ſubjects, a perfect accuracy is not to be expected. This learned gentleman himſelf allows, that, the conſidering of all the particles (or the ring of moons) as being in the ſame plane, produces no error in the concluſion: from whence it might, with ſome reaſon, be imagined, that the hypotheſis itſelf could not be otherwiſe than true. And it ſeems farther plain to me, that, whatever lights that Author's overſights in the ſolution of this Problem are capable of being placed, his real miſtakes are two only.

Now, according to this way of proceeding, it will be found, that the momentum of the whole earth (taken as a ſphere) will be to the momentum of a very ſlender ring, of the ſame diameter, revolving in the ſame time, about the ſame axe, in a proportion compounded of the proportion of the matter in the earth to the matter in the ring, and of the number 800000 to the number 1000000. Which proportion, therefore, differs from *that* of Sir ISAAC NEWTON, given above, in the ratio of 800000 to 925725: ſo that, if his reſult, which is $9'' 7''' 20^{\text{iv}}$, be increaſed in this ratio, we ſhall then have $10'' 33'''$, for the quantity of the annual preceſſion of the equinox, ariſing from the force of the ſun; allowing for the above-mentioned difference.

It appears further, by peruſing his 39th Propoſition, that he there aſſumes it as a principle, that, if a ring, encompaſſing the earth, at its equator AI*a*L *(but detached therefrom)* was to tend, or begin, to move about its diameter LI with the ſame accelera-

tive

Fig. 8.

tive force, or angular celerity, as *that* whereby the earth itſelf tends to move about the ſame diameter, through the action of the ſun (at S); that *then* the motion of the nodes of the ring and of the equator would be exactly *the ſame*. Now this would indeed be the caſe, were not the effects of theſe forces whereby the two bodies tend to move about the diameter LI, to be influenced and interrupted by the other motions about the axe of rotation P*p*, and that according to a different ratio, depending on the different figures of the earth and ring.

A ſphere, let the direction of its rotation be which way it will, that is, let it move about what diameter it will, has always the *ſame momentum*, provided it has the ſame angular celerity: but the *momentum* of a very ſlender ring, revolving about one of its diameters, appears *(by Lem. IV.)* to be only the half of what it would be, if the revolution was to be performed in a plane, about the center of the ring. Whence it is evident, that the ring AI*a*L, to acquire the ſame motion of preceſſion and nutation with the earth's equator, ought to tend to move about the diameter LI with an accelerative force double to *that* whereby the earth itſelf tends to move about the ſame diameter, through the action of the ſun: ſince, in this caſe, the quantities of motion, or the *momenta*, generated in the two bodies, during any very ſmall particle of time, would be exactly proportional to the reſpective *momenta* of rotation, whereby the bodies endeavour to perſevere in their preſent ſtate and direction of motion, in oppoſition to any new force impreſſed. Hence it follows that all concluſions, relating to the change of the poſition of the earth's axe, drawn from the principle above ſpecified, muſt be too little by juſt one half; and conſequently that the quantity of the annual preceſſion of the equinox, ariſing from the action of the ſun, ought to be the double of $10'' \, 33'''$; which is $21'' \, 6'''$, and agrees, to a *third*, with what we before found it to be, by two different methods.

A

A very exact METHOD for finding the PLACE of a PLANET in its orbit, from a Correction of WARD's *hypotheſis*, by means of One, or more Equations, applied to the motion about the upper focus.

Fig. 18. LET ABPC be the ellipſes in which the planet revolves about the ſun in the lower focus S; let F be the upper focus, and M and *m* any two places of the planet indefinitely near to each other; and let FM, F*m*, SM, S*m* be drawn, as likewiſe MN, perpendicular to the greater axis AP: from the center F, with the radius FD $= 1$, let the circumference of a circle DE*e* be deſcribed, and from its interſection with FM draw EH perpendicular to AP: put AO $(= \mathrm{OP}) = a$, OB $(= \mathrm{OC}) = b$, OF $(= \mathrm{OS}) = c$, FM $= u$, FH $= x$, EH $= y$, DE $= z$, and E*e* $= \dot{z}$: then SM being $(= \mathrm{AP} - \mathrm{FM}) = 2a - u$, by the nature of the ellipſes, and FN $(= \mathrm{FH} \times \frac{\mathrm{FM}}{\mathrm{FE}}) = xu$, we have, by a known property of triangles, $\overline{\mathrm{SM} + \mathrm{FM}} \times \overline{\mathrm{SM} - \mathrm{FM}}$ $(2a \times \overline{2a - 2u}) = 2\mathrm{SF} \times \mathrm{ON}$ $(4c \times \overline{c + xu})$: from which equation u (FM) is found $= \frac{a^2 - c^2}{a + cx} = \frac{bb}{a + cx}$ (becauſe $bb = aa - cc$): whence SM $(= 2a - u)$ is alſo had in terms of x, being $= \frac{aa + cc + 2acx}{a + cx}$.

Now the area EF*e* being expreſſed by $\frac{1}{2}\dot{z}$ $(= \mathrm{FE} \times \frac{1}{2}\mathrm{E}e)$, we have FE^2 (1) : FM^2 :: $\frac{1}{2}\dot{z}$: the area MF*m* $= \frac{1}{2}\dot{z} \times \overline{\mathrm{FM}}|^2$. Therefore, the angle SM*m* being equal to FMA, or F*m*A (by the nature of the curve) we alſo have FM×M*m* : S*m*×*m*M (or FM : SM) :: the area $(\frac{1}{2}\dot{z} \times \overline{\mathrm{FM}}|^2)$ of the triangle FM*m* : the area of the triangle SM*m* $= \frac{1}{2}\dot{z} \times \mathrm{FM} \times \mathrm{SM}$ (*Elem.* 23. 6.) $= \frac{\frac{1}{2}b^2\dot{z} \times \overline{aa + cc + 2acx}}{\overline{a + cx}|^2} = \frac{1}{2}b^2\dot{z} + \frac{\frac{1}{2}b^2c^2\dot{z} \times \overline{1 - xx}}{\overline{a + cx}|^2} = \frac{1}{2}b^2\dot{z} + \frac{\frac{1}{2}b^2c^2y^2\dot{z}}{\overline{a + cz}|^2}$ = the fluxion of the area ASM. In order to find the fluent hereof, let $\frac{1}{\overline{a+cx}|^2}$ be reſolved into the ſeries $\frac{1}{a^2} - \frac{2cx}{a^3}$, *&c.* (whereof the two firſt terms will, here, be ſufficient) by which means our fluxion will become $\frac{1}{2}b^2\dot{z} + \frac{\frac{1}{2}b^2c^2y^2\dot{z}}{a^2} - \frac{b^2c^3x\dot{z}y^2}{a^3}$; which, becauſe $y\dot{z} = -\dot{x}$, and $x\dot{z} = \dot{y}$, will be reduced to $\frac{1}{2}b^2\dot{z} + \frac{b^2c^2}{2a^2} \times -\dot{x}y - \frac{b^2c^3y^2\dot{y}}{a^3}$; whoſe fluent will therefore be truly expreſſed by $\frac{1}{2}b^2z + \frac{b^2c^2}{2a^2} \times \text{area DHE} - \frac{b^2c^3y^3}{3a^3}$, or $\frac{1}{2}b^2z + \frac{b^2c^2}{2a^2} \times \overline{\frac{1}{2}z - \frac{1}{2}xy} - \frac{b^2c^3y^3}{3a^3}$ (becauſe the area DHE $= \frac{1}{2}\text{DE} \times \text{DF} - \frac{1}{2}\text{HF} \times \text{EH} = \frac{1}{2}z - \frac{1}{2}xy$). This expreſſion, when M coincides with P, and z is = the ſemi-circumference DEK ($= p$), will become $\frac{b^2p}{2} + \frac{b^2c^2p}{4aa} = \frac{b^2dp}{2}$ (ſuppoſing $d = 1 + \frac{cc}{2aa}$) = the area of the ſemi-ellipſis ABP. Therefore we have, as $\frac{b^2dp}{2} : \frac{b^2dz}{2} - \frac{b^2c^2xy}{4aa} - \frac{b^2c^3y^3}{3a^3}$ (= the area ASM) :: p (= the length of an arch of 180°) : $z - \frac{c^2xy}{2a^2d} - \frac{2c^3y^3}{3a^3d}$ = the length of an arch (A) expreſſing the planet's mean anomaly: from which equation, $z = \text{A} + \frac{c^2xy}{2a^2d} + \frac{2c^3y^3}{3a^3d} = \text{A} + \frac{c^2}{4a^2d} \times \overline{\text{ſin. }2z} + \frac{2c^3}{3a^3d} \times \overline{\text{ſin. }z}|^3$ (becauſe xy, or co-ſin. $z \times$ ſin. z = $\frac{1}{2}$ ſin. $2z$): where the two laſt terms being very ſmall in compariſon of the others (and, therefore, z nearly = A), we may, inſtead of ſin. z and ſin. $2z$, ſubſtitute ſin. *A* and ſin. 2*A*; by which means we have $z = \text{A} + \frac{cc}{4a^2d} \times \text{ſin. }2\text{A} + \frac{2c^3}{3a^3d} \times \overline{\text{ſin. A}}|^3$. From whence it appears, that, in order to have the angle AFM at the upper focus, the mean anomaly (A) of the planet at the time given, muſt be increaſed by the quantity, or correction $\frac{cc}{4a^2d} \times \text{ſin. }2\text{A} + \frac{2c^3}{3a^3d} \times \overline{\text{ſin. A}}|^3$.

But to expreſs the value of this correction in ſeconds of a degree (which in practice is the moſt commodious) it will be, as 3,1416 (the length of an arch of 180 degrees) is to 648000 (the number of ſeconds in that arch) ſo is $\frac{cc}{4a^2d} \times \text{ſin. } 2A + \frac{2c^3}{3a^3d} \times \overline{\text{ſin. A}}|^3$ to $51567 \times \frac{c^2}{a^2d} \times \text{ſin. } 2A + 137513 \times \frac{c^3}{a^3d} \times \overline{\text{ſin. A}}|^3 =$ the number of ſeconds in the ſaid correction: the logarithm of the latter term of which will, therefore, be $= 5{,}1383 - \text{log. } d + 3 \text{ log. } \frac{c}{a} + 3 \text{ log. ſin. A}$; and that of the former $= 4{,}7123 - \text{log. } d + 2 \text{ log. } \frac{c}{a} + \text{log. ſin. } 2A$. But, to render theſe expreſſions ſtill more convenient for practice, the log. of d, by reaſon of its ſmallneſs, may be, either, intirely neglected, or elſe ſo aſſumed, to be nearly a mean of what it is known to be in the planetary orbits. Accordingly, by aſſuming the excentricity $c = \frac{8}{100}$ of the mean diſtance (which is a ſmall matter leſs than the excentricity of *Mars*, but ſomething greater than *thoſe* of the *Moon*, *Saturn*, and *Jupiter)*, the value of d $(= 1 + \frac{cc}{2aa})$ will be $= 1{,}0032$, and its logarithm $= 0{,}0014$. Whence the log. of the former part of our correction, by ſubſtituting this value, will be $5{,}1369 + 3 \text{ log. } \frac{c}{a} + 3 \text{ log. ſin. A} = 3 \times \overline{1{,}7123 + \text{log. } \frac{c}{a} + \text{log. ſ. A}}$; and *that* of the latter $= 4{,}7109 + 2 \text{ log. } \frac{c}{a} + \text{log. ſin. } 2A$: which, expreſſed in words at length, give the following

Practical Rules.

1°. To the ſum of the conſtant logarithm 1,7123 and the log. of the excentricity in parts of the mean diſtance, add the log. ſine of the mean anomaly; the ſum (rejecting the radius) being tripled, will give the log. of the firſt equation (in ſeconds) to be added to the mean anomaly.

2°. To the ſum of the conſtant log. 4,7109 and twice the log. of the excentricity, add the log. ſine of twice the mean anomaly; the ſum (rejecting the radius) will be the log. of the ſecond equation; to be added or ſubtracted, according as the mean anomaly is leſs, or greater than 90 degrees.

Here, and in what follows, the anomaly is to be always reckoned from, or to the aphelion, the neareſt way; in which the ſeconds may be omitted, in computing the propoſed corrections. Which corrections being made, the true anomaly, or angle at the lower focus, will be had from the common proportion, by ſaying, as the aphelion-diſtance is to the perihelion-diſtance, ſo is the tangent of half the mean anomaly, thus corrected, to the tangent of half the true anomaly ſought.

As an example of the uſe of the method here laid down, let the excentricity be ſuppoſed = 0,048219 (being that aſſigned, by Dr. HALLEY, to the orbit of Jupiter) and let the mean anomaly be 45°.

Then, log. excent. . .	$\overline{2},6832$	2 log. excent. . . .	3,3664
+ conſt. log. . .	1,7123	+ conſt. log. . . .	4,7109
= log. for 1ſt equ.	0,3955	= log. for 2d equ.	2,0773
log. ſin. anom. . .	9,8494	log. ſin. 2 anom.	10,0000
	0,2449	2d equ. = $119\frac{1}{2}''$	2,0773
firſt equ. = $5\frac{1}{2}''$	0,7347		

From $\overline{1},978537$ = log. of perihelion-diſt. 0,951781,
ſubtr. 0,020452 = log. of the aphelion-diſt. 1,048219;
the rem. $\overline{1},958085$, will be a (3d) conſt. log. for this orbit: to which
add 9,617596 = log. tang. $\frac{1}{2}$ cor. anom. 22° 31′ $2\frac{1}{2}''$,
ſo ſhall 9,575681 = log. tang. $\frac{1}{2}$ true anomaly 20° 37′ 40″: whoſe double, 41° 15′ 20″, is therefore the true anomaly required.

The ſame excentricity being retained; let the mean anomaly be, now, ſuppoſed 120 degrees.

Then, log. ſin. anom.	9,9375	log. ſin. 2 anom. . .	9,9375
log. for firſt equat.	0,3955	log. for 2d equation	2,0773
	0,3330	2d equat. = $103\frac{1}{2}''$	2,0148
firſt equat. = 10″	0,9990		

Hence 120° + 10″ — 1′ $43\frac{1}{2}''$ = 119° 58′ $26\frac{1}{2}''$ = the cor. anomaly.

Therf. log. tang. $\frac{1}{2}$ cor. anom. 59° 59′ $13\frac{1}{2}''$ . . . 10,238331
+ third conſt. log. for this orbit . . . . $\overline{1},958085$
= log. tang. $\frac{1}{2}$ true anom. 57° 32′ 10″ 10,196416

Whence the anomaly itſelf is given $= 3^{s}\,25^{\circ}\,4'\,20''$.

If the excentricity be aſſumed $= 0{,}066777$ (being the greateſt that Dr. HALLEY gives to the lunar orbit) the two conſtant logarithms, to be added to the ſines of the mean anomaly and of its double, will be 0,5369 and 2,3601: whence if the ſaid anomaly be taken $= 50^{\circ}$ (at which, according to the *Doctor's Table, pro expediendo calculo æquationis centri lunæ*, the whole correction is a *maximum)*, the former part of the ſaid correction will be found $= 18\tfrac{1}{2}''$, and the latter part $= 225\tfrac{1}{2}''$: therefore the ſum of both is $244''$, or $4'\,4''$; agreeing, exactly, with the quantity given in the *Table*. And in the very ſame manner, the proper corrections correſponding to other anomalies and excentricities may be computed; the error never amounting to above a ſingle ſecond in any of the planets, except *Mars* and *Mercury:* in the place of *Mars*, the greateſt error will be two or three ſeconds; and in that of *Mercury*, about as many minutes.—As to the *Earth* and *Venus*, the ſecond equation, alone, will be ſufficient to give their places to leſs than a ſecond.

To obtain a farther correction, which will be neceſſary when the orbit is very eccentrical, we may (inſtead of the two firſt only) make uſe of a greater number of terms of the ſeries $\frac{1}{a^{2}} - \frac{2cx}{a^{3}} + \frac{3c^{2}x^{2}}{a^{4}} - \frac{4c^{3}x^{3}}{a^{5}}$ *&c.* $\left(= \frac{1}{\overline{a+cx}|^{2}}\right)$; by which means the fluxion $\left(\tfrac{1}{2}b^{2}\dot{z} + \frac{\frac{1}{2}b^{2}c^{2}y^{2}\dot{z}}{\overline{a+cx}|^{2}}\right)$ of the area MSA deſcribed about the focus S, which is proportional to the time, will be here repreſented by $\tfrac{1}{2}b^{2}\dot{z} + \tfrac{1}{2}b^{2} \times \overline{e^{2}y^{2}\dot{z} - 2e^{3}y^{2}x\dot{z} + 3e^{4}y^{2}x^{2}\dot{z} - 4e^{5}y^{2}x^{3}\dot{z}}$ *&c.* (ſuppoſing $e = \frac{c}{a}$). But it is well known that y^{2} $(\overline{\text{ſin.}\ z}|^{2})$ $= \tfrac{1}{2} - \tfrac{1}{2}$ co-ſin. $2z$; whence $y^{2}x$ $(= y^{2} \times$ co-ſin. $z)$ $= \tfrac{1}{2}$ co-ſin. $z - \tfrac{1}{2}$ co-ſin. $2z \times$ co-ſin. $z = \tfrac{1}{2}$ co-ſin. $z - \tfrac{1}{4}$ co-ſin. $z - \tfrac{1}{4}$ co-ſin. $3z$* $= \tfrac{1}{4}$ co-ſin. $z - \tfrac{1}{4}$ co-ſin. $3z$; and therefore $y^{2}x^{2}$

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* This, and all that follows to the ſame purpoſe, is nothing more than the application of the THEOREM, *That the rectangle of the co-ſines of any two angles (the radius being unity) is equal to half the ſum of the co-ſines of the ſum and difference of thoſe angles.*

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$= \frac{1}{4}\,\text{co-ſin.}\,z \times \text{co-ſin.}\,z - \frac{1}{4}\,\text{co-ſin.}\,3z \times \text{co-ſin.}\,z = \frac{1}{8} + \frac{1}{8}\,\text{co-ſin.}\,2z - \frac{1}{8}\,\text{co-ſin.}\,2z - \frac{1}{8}\,\text{co-ſin.}\,4z = \frac{1}{8} - \frac{1}{8}\,\text{co-ſin.}\,4z$: whence alſo $y^2x^3 = \frac{1}{8}\,\text{co-ſin.}\,z - \frac{1}{8}\,\text{co-ſin.}\,4z \times \text{co-ſin.}\,z = \frac{1}{8}\,\text{co-ſin.}\,z - \frac{1}{16}\,\text{co-ſin.}\,3z - \frac{1}{16}\,\text{co-ſin.}\,5z$. Which values being now ſubſtituted for their equals, our fluxion, above given, will become $\frac{1}{2}b^2\dot{z} + \frac{1}{2}b^2$ into $e^2 \times \overline{\frac{1}{2}\dot{z} - \frac{1}{4} \times 2\dot{z}\,\text{co-ſin.}\,2z} - 2e^3 \times \overline{\frac{1}{4}\dot{z}\,\text{co-ſ.}\,z - \frac{1}{12} \times 3\dot{z}\,\text{co-ſ.}\,3z} + 3e^4 \times \overline{\frac{1}{8}\dot{z} - \frac{1}{32} \times 4\dot{z}\,\text{co-ſ.}\,4z} - 4e^5 \times \overline{\frac{1}{8}\dot{z}\,\text{co-ſ.}\,z - \frac{1}{48} \times 3\dot{z}\,\text{co-ſ.}\,3z - \frac{1}{80} \times 5\dot{z}\,\text{co-ſ.}\,5z} + \&c.$ and, conſequently, the fluent thereof $= \frac{1}{2}b^2z + \frac{1}{2}b^2$ into $e^2 \times \overline{\frac{1}{2}z - \frac{1}{4}\,\text{ſin.}\,2z} - 2e^3 \times \overline{\frac{1}{4}\,\text{ſ.}\,z - \frac{1}{12}\,\text{ſ.}\,3z} + 3e^4 \times \overline{\frac{1}{8}z - \frac{1}{32}\,\text{ſin.}\,4z} - 4e^5 \times \overline{\frac{1}{8}\,\text{ſine}\,z - \frac{1}{48}\,\text{ſine}\,3z - \frac{1}{80}\,\text{ſine}\,5z}\ \&c. = \frac{1}{2}b^2$ into $\overline{1 + \frac{1}{2}e^2 + \frac{3}{8}e^4} \times z - \overline{\frac{1}{2}e^3 + \frac{1}{2}e^5} \times \text{ſin.}\,z - \frac{1}{4}e^2\,\text{ſin.}\,2z + \overline{\frac{1}{6}e^3 + \frac{1}{12}e^5} \times \text{ſin.}\,3z - \frac{3e^4}{32}\,\text{ſin.}\,4z + \frac{e^5}{20}\,\text{ſin.}\,5z$ (ſuppoſing all ſuch terms wherein e riſes to more than five dimenſions, to be diſregarded). This expreſſion, when $z = p$ (= the ſemi-circumference AEK) becomes $\frac{1}{2}b^2 \times \overline{1 + \frac{1}{2}e^2 + \frac{3}{8}e^4\ \&c.} \times p = \frac{1}{2}b^2p \times \overline{1 - ee}|^{-\frac{1}{2}} = \frac{\frac{1}{2}b^2p}{\sqrt{1 - \frac{cc}{aa}}} = \frac{\frac{1}{2}b^2p \times a}{\sqrt{a^2 - c^2}} = \frac{1}{2}abp$ = the area of the ſemi-ellipſe ABP. Hence it will be, as $\frac{1}{2}abp$ (area ABP) : $\frac{1}{2}abz - \frac{1}{2}b^2 \times \overline{\frac{1}{2}e^3 + \frac{1}{2}e^5} \times \text{ſin.}\,z - \frac{1}{2}b^2 \times \frac{1}{4}e^2\,\text{ſin.}\,2z\ \&c.$ (area ASM) : : p (the length of an arch of 180°) : $z - \frac{b}{a} \times \overline{\frac{1}{2}e^3 + \frac{1}{2}e^5} \times \text{ſin.}\,z - \frac{b}{a} \times \frac{e^2}{4}\,\text{ſin.}\,2z\ \&c.$ = the length of the arch A, expreſſing the planet's mean anomaly: whence, by ſubſtituting $f = \frac{b}{a}$, we have $A = z - \overline{1 + e^2} \times \frac{1}{2}fe^3\,\text{ſin.}\,z - \frac{1}{4}fe^2\,\text{ſin.}\,2z + \overline{1 + \frac{1}{2}e^2} \times \frac{1}{6}fe^3\,\text{ſin.}\,3z - \frac{3fe^4}{32} \times \text{ſin.}\,4z + \frac{fe^5}{20} \times \text{ſine}\,5z$.

Now, to find from hence the value of z, in terms of A and the ſines of its multiples, we may, for a firſt approximation, aſſume $\overline{1 + e^2} \times \frac{1}{2}fe^3\,\text{ſin.}\,z + \frac{1}{4}fe^2\,\text{ſin.}\,2z\ \&c.$ as equal to $\overline{1 + e^2} \times \frac{1}{2}fe^3\,\text{ſin.}\,A + \frac{1}{4}fe^2\,\text{ſin.}\,2A - \overline{1 + \frac{1}{2}e^2} \times \frac{1}{6}fe^3\,\text{ſin.}\,3A\ \&c.$ which laſt quantity being denoted by Q, we ſhall have $A = z - Q$, and conſequently $z = A + Q$.

But ſin. z ($=$ ſin. $\overline{\mathrm{A}+\mathrm{Q}}$) $=$ ſin. A × co-ſin. Q $+$ ſin. Q × co-ſin. A $=$ ſin. A $+$ Q × co-ſin. A, nearly (becauſe Q being very ſmall, its co-ſine will differ inſenſibly from the radius, and the ſine very little from the arch itſelf). In the ſame manner, ſ. $2z$ ($=$ ſ. $\overline{2\mathrm{A}+2\mathrm{Q}}$) $=$ ſ. 2A $+$ 2Q × co-ſ. 2A, ſin. $3z$ $=$ ſ. 3A $+$ 3Q × co-ſ. 3A, &c. Which values being therefore ſubſtituted in the given equation, it will become A $= z -$ Q $-$ Q × $\overline{\overline{1+e^2}\times\frac{1}{2}fe^3\,\text{co-ſ.A}+\frac{1}{2}fe^2\,\text{co-ſ.}\,2\text{A}-\overline{1+\frac{1}{2}e^2}\times\frac{1}{2}fe^3\,\text{co-ſ.}\,3\text{A}}$ &c. But the firſt term of the ſeries, $\overline{1+e^2}\times\frac{1}{2}fe^3$ co-ſin. A, drawn into $-$ Q (if all terms having more than five dimenſions of e be neglected) will be barely $= \frac{1}{2}fe^3 \times$ co-ſin. A $\times -\frac{1}{4}fe^2$ ſin. 2A $= -\frac{1}{8}f^2e^5 \times$ co-ſin. A × ſin. 2A $= -\frac{1}{16}f^2e^5 \times \overline{\text{ſin. }3\text{A}+\text{ſin. A}}$.

In the ſame manner $\frac{1}{2}fe^2$ co-ſin. 2A $\times -$ Q $= -\frac{1}{2}fe^2$ co-ſin. 2A $\times \overline{\frac{1}{2}fe^3\,\text{ſin. A}+\frac{1}{4}fe^2\,\text{ſin. }2\text{A}-\frac{1}{6}fe^3\,\text{ſin. }3\text{A}} = -\frac{1}{8}f^2e^5 \times \overline{\text{ſin.}\,3\text{A}-\text{ſin. A}} - \frac{1}{16}f^2e^4 \times$ ſin. 4A $+ \frac{1}{24}f^2e^5 \times \overline{\text{ſin. }5\text{A}+\text{ſin. A}}$: And, laſtly, $-\overline{1+\frac{1}{2}e^2} \times \frac{1}{2}fe^3$ co-ſin. 3A $\times -$ Q equal to $\frac{1}{2}fe^3$ co-ſin. 3A × Q $= \frac{1}{2}fe^3 \times$ co-ſin. 3A $\times \frac{1}{4}fe^2$ ſin. 2A $= \frac{1}{16}f^2e^5 \times \overline{\text{ſin. }5\text{A}-\text{ſin. A}}$. The ſum of all which will be $\frac{1}{24}f^2e^5$ ſin. A $- \frac{3}{16}f^2e^5$ ſin. 3A $- \frac{1}{16}f^2e^4$ ſin. 4A $+ \frac{5}{48}f^2e^5$ ſin. 5A; which added to $z -$ Q (or its equal, $z - \overline{1+e^2} \times \frac{1}{2}fe^3$ ſin. A $- \frac{1}{4}fe^2$ ſin. 2A, &c.) gives A $= z - \overline{1+e^2-\frac{1}{12}fe^2} \times \frac{1}{2}e^3f$ ſin. A $- \frac{1}{4}e^2f$ ſin. 2A $+ \overline{1+\frac{1}{2}e^2-\frac{9}{8}e^2f} \times \frac{1}{6}e^3f$ ſin. 3A $- \overline{3f+2f^2} \times \frac{e^4}{32}$ ſin. 4A $+ \overline{\frac{1}{20}f+\frac{5}{48}f^2}\,e^5 \times$ ſin. 5A. From whence we have $z =$ A $+ \overline{1+\frac{11ee}{12}} \times \frac{1}{2}e^3f$ ſin. A $+ \frac{1}{4}e^2f$ ſin. 2A $- \overline{1-\frac{5ee}{8}} \times \frac{1}{6}e^3f$ ſin. 3A $+ \frac{5e^4f^2}{32}$ ſin. 4A $- \frac{37e^5f}{240}$ ſin. 5A, very near; becauſe in all terms having more than three dimenſions of e, the quantities 1 and f may be uſed indifferently, for each other, without producing any errors but ſuch as conſiſt of more than five dimenſions of the converging quantity e.

But, ſince it is known that ſin. 3A $= 3\text{S} - \text{S}^3$, and ſin. 5A $= 5\text{S} - 20\text{S}^3 + 16\text{S}^5$ (S being the ſine of A) we ſhall, by ſubſtituting theſe values in the 2d, 4th, and 6th terms (after proper reduction) have $z = A + \overline{1 + 4ee} \times \frac{2}{3}e^3 f S^3 - \frac{37e^5 f}{15} S^5 + \frac{1}{4}e^2 f \times \text{ſin.}\, 2A + \frac{5e^4 f^2}{32} \times \text{ſin.}\, 4A$. Hence it appears that $\frac{180 \times 60 \times 60}{3{,}1416} \times \overline{\overline{1 + 4ee} \times \frac{2}{3}e^3 f S^3 - \frac{37e^5 f}{15} S^5 + \frac{1}{4}e^2 f \,\text{ſ.}\, 2A + \frac{5e^4 f^2}{32} \text{ſ.}\, 4A}$ will expreſs the number of ſeconds, to be added to the mean anomaly, in order to have the angle AFM at the upper focus of the ellipſis, correſponding to that anomaly. From whence is deduced the following method of calculation.

Let *F* denote the logarithm of half the leſser axis divided by half the greater, *E* the log. of the eccentricity divided by half the greater axis, *G* the log. of the ſum of the ſquares of half the greater axis and twice the eccentricity divided by the ſquare of half the greater axis.

$$\text{Take } P = 1{,}71277 + E + \tfrac{1}{3}F + \tfrac{1}{3}G,$$
$$Q = 1{,}14130 + E + \tfrac{1}{5}F,$$
$$R = 4{,}71236 + 2E + F,$$
$$S = 4{,}50824 + 4E + 2F;$$

then the logarithms of four equations (in ſeconds), to be applied to the mean anomaly (A), will be

$$3 \times \overline{P + \text{log. ſin. } A - \text{log. rad.}}$$
$$5 \times \overline{Q + \text{log. ſin. } A - \text{log. rad.}}$$
$$R + \text{log. ſin. } 2A - \text{log. rad.}$$
$$S + \text{log. ſin. } 4A - \text{log. rad.}$$

Of which equations the firſt is always to be added, and the ſecond always ſubtracted; the other two being to be added, or ſubtracted, according as the ſines of their reſpective arguments, 2A and 4A, are poſitive, or negative.

The two principal of theſe equations agree with thoſe before given; and are the ſame, in effect, with the two equations laid down (without demonſtration) by Sir ISAAC NEWTON, in the Scholium to the 31ſt Propoſition of the firſt book of his *Principia*. The latter of which, in the *Laws of the Moon's Motion*, prefixed to that Work, ſeems to be repreſented, as *defective*; it being there aſſerted, that, *the inequality in the motion of a planet about the upper focus, conſiſts of three parts*; as if the nature of the ſubject admitted of juſt that number, and no more; whereas the parts, or equations ariſing in the conſideration, are without number, being the terms of an infinite ſeries, wherein the eccentricity is the converging quantity. Sir ISAAC NEWTON has given two terms of this ſeries, which are right; but the new equation added by his *Commentator*,, is not ſo; the ſign thereof, the coefficient, and the law by which it increaſes and decreaſes, being all different from what they ought to be.

This equation (expreſſed according to the above notation, where 1 repreſents half the greater axis, *f* half the leſſer axis, *e* the eccentricity, and *A* the mean anomaly) he makes to be $-\frac{1}{2}e^4 \times \overline{\text{ſin. A}}|^3 \times$ co-ſin. A. But it ought to be $+\frac{3f+2f^2}{32} \times e^4$ ſin. 4A ($= \frac{5e^4}{32} \times$ ſin. 4A, nearly), as is ſhewn above; this being the 3d term of the general ſeries, and the next in order after thoſe given by Sir ISAAC NEWTON; who appears, more than once*, to have been diſadvantageouſly (I might ſay, unfairly) repreſented, and that, under the covers of his own book: a circumſtance that cannot be attended to without ſome concern and diſlike, by thoſe who entertain a due regard for the merit of an Author to whom the mathematical world is ſo much indebted.

I ſhall now put down one ſingle example of the uſe of the equations above derived; wherein I ſhall ſuppoſe the eccentricity to be $\frac{20589}{100000}$ parts of the ſemi-tranſverſe axis (the ſame as is aſſigned by Dr. HALLEY to the orbit of Mercury). Here, then, we have $E = -\overline{1},313635$; F ($= \log.\sqrt{1-ee} = \frac{1}{2}\log.\overline{1-ee}$) $= -0,009406$; G ($= \log.\overline{1+4ee}$) $= 0,068024$; whence $P = 1,046$; $Q = 0,453$; $R = 3,3302$; $S = 1,744$:

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* In the 28th Propoſition of his third book, it is found that the moon's diſtance from the earth in the ſyzigies is to the diſtance in the quadratures (ſetting aſide the conſideration of eccentricity) as 69 to 70; which is confirmed by what is demonſtrated in a ſubſequent part of this our Work, as well as by the calculations of others; nevertheleſs the truth of this proportion is called in queſtion, and a new one is laid down, which makes the ſaid diſtances to be in the proportion of 59 to 6c. See *Laws of the Moon's Motion*, p. 11 and 12.

which values will ſerve in all caſes belonging to this orbit. Whence, ſuppoſing the given anomaly to be 50°, we have

$$\begin{array}{rrrr}
1{,}0460 & 0{,}4530 & 3{,}3302 & 1{,}744 \\
9{,}8842 & 9{,}8842 & 9{,}9933 & 9{,}534 \\
\hline
0{,}9302 & 0{,}3372 & 3{,}3235 & 1{,}278. \\
3 & 5 & & \\
\hline
2{,}7906 & 1{,}6860 & &
\end{array}$$

Of which reſulting logarithms, the numbers correſponding are $617\frac{1}{2}$, $48\frac{1}{2}$, $2106\frac{1}{4}$, and 19; whereof the firſt and third being added, and the other two ſubtracted, we have 50° 44′ 16″ for the corrected anomaly, or the angle at the upper focus; whereof the half is 25° 22′ 8″:

Therefore log. tang. 25° 22′ 8″ 9,6759338
+ log. of the ratio of the gr. and leaſt diſt. 1,8185730
= log. tang. $\frac{1}{2}$ true anomaly 17° 20′ 28″ 9,4945068.

Which concluſion is true to a ſecond. Nor will the error, in any part of this orbit, amount to more than about two or three ſeconds.—If you would have the reſult depended on to a ſingle ſecond, or if the orbit be ſuppoſed ſtill more eccentrical than that of Mercury, then the following method may be of uſe.

Say, as half the greater axis of the ellipſis is to the eccentricity, ſo is the ſine of the mean anomaly to the ſine of an angle; which ſubtract from the mean anomaly, and to the log. ſine of the remainder (which I call the eccentric anomaly) add the ſum of the log. of the eccentricity and the conſtant log. 1,758123: the aggregate (rejecting the radius) will be the logarithm of an angle, in degrees and decimal parts; which, ſubtracted from the angle firſt found, leaves a correction to be added (under its proper ſign) to the mean anomaly: with which corrected anomaly, let the whole operation be repeated, if needful, by always adding the laſt correction to the mean anomaly. Then it will be, as the greater ſemi-axis of the ellipſis is to the leſſer, ſo is the tangent of the corrected anomaly to the tangent of the angle at the upper focus of the ellipſis: whence the angle at the lower focus, or the true anomaly, may alſo be known by the common proportion, and that to any aſſigned degree of exactneſs.

This method, in all the planets, except *Mars* and *Mercury*, anſwers to a ſecond, at one operation. In the former of theſe two, the error, when greateſt, will amount to about three or four ſeconds; and in the latter, to nearly as many minutes; in which caſe, three operations will be neceſſary: but in order to avoid that trouble, the following calculation may be uſed; which is ſo exact as to anſwer, even in the orbit of Mercury, to leſs than half a ſecond, without repeating the operation.

Add together twice the log. of the eccentricity, the log. ſecant of the angle firſt found (as above), and the log. co-ſine of the ſum of the eccentric anomaly and mean anomaly once corrected (in all of which angles the ſeconds may be neglected). The aggregate (ſubtracting twice the radius) will be the log. of a fraction to be added to unity, when the ſaid ſum of anomalies is between 90 and 270°; but otherwiſe, ſubtracted therefrom: then the log. of this ſum, or remainder being ſubtracted from the log. of the firſt correction, you will have the log. of the true correction to be added (under its proper ſign) to the mean anomaly given.

Thus, for example, let the mean anomaly be 70; and let the eccentricity be = 0,20589 (as in the preceding example).

Here, ſin. mean anom. 70° 9,9729858
log. eccent. 1,3136353
∴ ang. firſt found = 11° 9′,33 9,2866211;

whence 58° 50′,67 = the eccentric anomaly:

whoſe ſine 9,9323552
+ the log. proper for the orbit . . 1,0717583
= log. of 10°,0952 . . . . 1,0041135:

which angle, or its equal, 10° 5′,71, being ſubtracted from 11° 9′,33, leaves the firſt correction 1° 3′,62 = 63′,62.

Moreover, by adding together 70°, 1° 3′, and 58° 50′, we have

129° 53′, whoſe co-ſine 9,8070
+ ſec. 11° 9′ ang. firſt found . . . 10,0083
+ twice log. eccent. . . . . $\bar{2}$,6272
= log. of 0,0272 . . . . $\bar{2}$,4425

Log. firſt cor. 63′,62 1,80359
— log. 1,0272 0,01187
= log. true cor. 61′,9 . . . 1,79172

Now the mean anom. + true cor. = 71° 1′,9, whoſe log. tang. 10,4638087
+ log. of the ſemi-conjugate axis . . . . . . $\bar{1}$,9905940
= log. tang. of 70° 38′,82 10,4544027

And the log. tang. of the half hereof, 35° 19′,41 9,8504350
+ log. of the ratio of the gr. and leaſt diſtances $\bar{1}$,8185730
= log. tang. of 25° 1′,02 9,6690080

whoſe double 50° 2′,04, or 50° 2′ 2″, is the true anomaly required.

A

DETERMINATION OF THE Difference between the MOTION of a COMET in an Elliptic, and a Parabolic Orbit.

Fig. 19. LET PNG be a parabola, and PBH a very excentrical ellipſis, having the ſame focus S, and vertex P with the parabola; let moreover N and *n* be conſidered as cotemporary poſitions of two bodies, in theſe orbits, moving from the perihelion P at the ſame time, about the ſun in the focus S. Make NBC perpendicular to PSO, and call PC, *x*; PS, *c*; and the greater axis of the ellipſis, *a*: then the leſſer axis will be $= 2\sqrt{c \times \overline{a - c}}$; the parameter $= \frac{4ac - 4cc}{a}$; and the ordinate BC $= 2\sqrt{\frac{\overline{ac - cc} \times \overline{ax - xx}}{aa}}$ (by the property of the ellipſis) $= 2c^{\frac{1}{2}}x^{\frac{1}{2}} \times \overline{1 - \frac{c}{a}}\big|^{\frac{1}{2}} \times \overline{1 - \frac{x}{a}}\big|^{\frac{1}{2}} = 2c^{\frac{1}{2}}x^{\frac{1}{2}} \times \overline{1 - \frac{c}{2a} - \frac{x}{2a}}$, nearly. This laſt taken from CN ($= 2c^{\frac{1}{2}}x^{\frac{1}{2}}$, by the nature of the parabola) leaves $c^{\frac{1}{2}}x^{\frac{1}{2}} \times \overline{\frac{c}{a} + \frac{x}{a}} =$ BN; which being multiplied by $\dot{x}$ and the fluent found, we thence have $c^{\frac{1}{2}}x^{\frac{1}{2}} \times \overline{\frac{2cx}{3a} + \frac{2x^2}{5a}}$ for the meaſure of the area NPB, or NP*v*, very near: which ſubtracted from the area NSP ($=$ CN $\times \frac{2}{3}$PC $-$ CN $\times \frac{1}{2}$CS $= \underline{2c^{\frac{1}{2}}x^{\frac{1}{2}} \times \frac{2}{3}x - 2c^{\frac{1}{2}}x^{\frac{1}{2}}} \times \overline{\frac{1}{2}x - \frac{1}{2}c} = c^{\frac{1}{2}}x^{\frac{1}{2}} \times \overline{c + \frac{1}{3}x}$), leaves $c^{\frac{1}{2}}x^{\frac{1}{2}} \times c + \frac{1}{3}x - \frac{2cx}{3a} - \frac{2xx}{5a} =$ the area *v*PS. Moreover, the parameter of the parabola being 4*c*, and

that of the ellipſis $= \frac{4ac - 4cc}{a}$, we have $\sqrt{4c}$ to $\sqrt{\frac{4ac - 4cc}{a}}$, ſo is the parabolic area NSP ($c^{\frac{1}{2}}x^{\frac{1}{2}} \times \overline{c + \frac{1}{3}x}$) to the correſponding, or cotemporary elliptical area nSP $= \sqrt{1 - \frac{c}{a}} \times c^{\frac{1}{2}}x^{\frac{1}{2}} \times \overline{c + \frac{1}{3}x}$ $= c^{\frac{1}{2}}x^{\frac{1}{2}} \times \overline{1 - \frac{c}{2a}} \times \overline{c + \frac{1}{3}x}$ (nearly) $= c^{\frac{1}{2}}x^{\frac{1}{2}} \times \overline{c + \frac{1}{3}x - \frac{cc}{2a} - \frac{cx}{6a}}$; which ſubducted from the area vPS, leaves $c^{\frac{1}{2}}x^{\frac{1}{2}} \times \overline{\frac{cc}{2a} - \frac{cx}{2a} - \frac{2xx}{5a}}$ $=$ the area vSn.

Let now NM be ſuppoſed perpendicular to the tangent NA, meeting the axis in M; then SM, SN, and SA will be all equal to each other, by the nature of the parabola; and conſequently the angle PMN equal to half PSN; whoſe tangent (to the radius 1) let be denoted by z: whence it will be, as $1 : z :: \text{MC}\,(2c) : \text{CN}\,(2c^{\frac{1}{2}}x^{\frac{1}{2}})$; and conſequently $x = cz^2$. Which value being ſubſtituted in the area vSn, it will become $= cz \times \overline{\frac{cc}{2a} - \frac{c^2z^2}{2a} - \frac{2c^2z^4}{5a}} = \frac{c^3z}{2a} \times \overline{1 - z^2 - \frac{4}{5}z^4}$; and this divided by $\frac{1}{2}\,\overline{\text{S}v}^2$ ($= \frac{1}{2} \times \overline{c + x}^2 = \frac{1}{2}c^2 \times \overline{1 + zz}^2$), gives $\frac{c^3z}{2a} \times \frac{1 - z^2 - \frac{4}{5}z^4}{\overline{1 + zz}|^2}$ for the meaſure of the angle vSn, in parts of the radius (1). Therefore, if m be put $= 3437 =$ the number of minutes in an arch equal to the radius, and $u = \frac{z \times \overline{1 - z^2 - \frac{4}{5}z^4}}{\overline{1 + zz}|^2}$, it is evident that $\frac{c}{a} \times mu$ will expreſs the meaſure of the ſaid angle in minutes of a degree. To find now the ratio Sn to SN, which ſtill remains to be determined, we have (by Trigonometry) $\text{N}v = \frac{\text{ſin. B}}{\text{ſin. B}v\text{F}} \times \text{BN} = \frac{\text{ſin. AFC}}{\text{ſin. AFS}} \times \text{BN} = \frac{\text{ſin. M}}{\text{co-ſin. M}}$ $\times \text{BN} = \text{tang. M} \times \text{BN} = z \times c^{\frac{1}{2}}x^{\frac{1}{2}} \times \frac{c + x}{a} = z \times cz \times \frac{c + czz}{a}$ $= \frac{cczz}{a} \times \overline{1 + zz}$. Alſo (ſuppoſing nb perpendicular to Sv) it will be, as 1 (radius) : $\frac{cu}{a}$ (the arch meaſuring the angle nSv)

$: : c \times \overline{1 + zz}$ ($=$ Sn, nearly) : $nb = \frac{ccu}{a} \times \overline{1 + zz}$: whence, again, by ſimilar triangles, $1 : z$ (: : CM : CN) : : $\frac{ccu}{a} \times \overline{1 + zz}$ (nb) : $bv = \frac{cczu}{a} \times \overline{1 + zz}$: this added to N$v = \frac{cczz}{a} \times \overline{1 + zz}$, gives N$b = \frac{ccz}{a} \times \overline{u + z} \times \overline{1 + zz}$: which therefore is to SN ($c \times \overline{1 + zz}$) as $\frac{cz}{a} \times \overline{u + z}$ to unity; and conſequently the required proportion of Sn (or Sb) to SN, as $1 - \frac{cz}{a} \times \overline{u + z}$ to unity.

From this laſt concluſion, and that derived above, exhibiting the value of the angle NSn, the following Table is computed; whoſe uſe is thus: Find, in the firſt column, the comet's longitude from the perihelion, as given from the hypotheſis of a parabolic orbit (either by Dr. HALLEY's Table, or any other of the like kind); againſt which, in the third column, you have the logarithm of a number of minutes (expreſſed in the ſecond column); from which ſubtracting the logarithm of the ratio of the greater axe of the ellipſe divided by the perihelion diſtance, the remainder will be the logarithm of a number of minutes to be added to, or ſubtracted from the aforeſaid longitude, as the Table directs: whereby the comet's longitude, for the ſame time, in the elliptic orbit will be given. And if, from the logarithm found in the fourth column, the logarithm of the ſame ratio be alſo ſubtracted, the remainder, abating 10, will be the logarithm of a quantity to be taken from the logarithm of the comet's diſtance from the ſun, computed according to the aforeſaid hypotheſis.

Thus, for example, let the greater axis of the ellipſis be ſuppoſed $=$ 35,727, and the perihelion diſtance $=$ 0,5825 (anſwering to the orbit of the comet of the year eighty-two); and let the longitude from the perihelion, according to Dr. HALLEY's Table, computed for a parabolic orbit, be 44° 3′ 20″; correſponding to which, the logarithm of the diſtance from the ſun is given $= 0{,}065838$. Here, then, the log. of $\frac{35{,}727}{0{,}5825}$ being $= 1{,}7877$, this value is to be ſubtracted from both the logarithms 2,9228 and 9,0555, ſtanding againſt 44°, in the third and fourth columns of the annexed Table: from whence will be found the two quantities 13′,65 and 0,001853; which being ſubtracted from 44° 3′ 20″, and 0,065838, the required longitude from the perihelion is given from thence = 43° 49′ 41″, and the logarithm of the comet's diſtance from the ſun = 0,063985.

The ſame Table, not only furniſhes an eaſy way, for deducing the motion in an elliptic orbit, from the motion in a parabolic one, but may be farther uſeful in determining, in ſome degree, the ſpecies of the ellipſis which a new comet deſcribes, when the obſervations thereon are found to differ ſenſibly from the places computed according to the hypotheſis of a parabolic orbit.

	ſubtr.	ſubtract	ſubtract		ſubtr.	ſubtract	ſubtract
1°	30	1,4770	5,8200	41°	836	2,9220	9,0006
2	60	1,7770	6,4220	42	836	2,9229	9,0195
3	90	1,9533	6,7735	43	836	2,9232	9,0378
4	119	2,0778	7,0242	44	836	2,9228	9,0555
5	149	2,1740	7,2178	45	835	2.9217	9,0727
6	179	2,2521	7,2759	46	832	2,9200	9,0894
7	208	2,3178	7·5095	47	827	2,9176	9,1056
8	237	2,3745	7,6250	48	821	2,9145	9,1215
9	266	2,4242	7,7269	49	814	2,9107	9,1369
10	294	2,4683	7,8178	50	806	2,9061	9,1520
11	322	2,5074	7,8998	51	796	2,9007	9,1666
12	349	2,5431	7,9747	52	784	2,8945	9,1808
13	376	2,5760	8,0437	53	772	2,8874	9,1947
14	403	2,6058	8,1074	54	758	2,8795	9,2083
15	430	2,6331	8,1666	55	743	2,8708	9,2215
16	455	2,6583	8,2217	56	726	2,8610	9,2344
17	480	2,6816	8,2735	57	708	2,8500	9,2469
18	505	2,7032	8,3222	58	688	2,8378	9,2592
19	529	2,7233	8,3682	59	667	2,8243	9,2712
20	552	2,7420	8,4116	60	645	2,8095	9,2829
21	575	2,7594	8,4528	61	621	2,7932	9,2943
22	596	2,7755	8,4921	62	596	2,7754	9,3055
23	617	2,7906	8,5295	63	570	2,7557	9,3163
24	637	2,8046	8,5652	64	542	2,7338	9,3270
25	657	2,8176	8,5994	65	512	2,7095	9·3374
26	676	2,8297	8,6320	66	482	2,6827	9,3477
27	694	2,8410	8,6634	67	450	2,6526	9,3576
28	710	2,8514	8,6934	68	416	2,6187	9,3674
29	726	2,8610	8,7224	69	381	2,5805	9,3769
30	741	2,8698	8,7502	70	344	2,5363	9,3862
31	755	2,8779	8,7770	71	306	2,4857	9,3954
32	768	2,8853	8,8030	72	267	2,4258	9,4044
33	780	2,8920	8,8280	73	226	2,3537	9,4132
34	791	2,8980	8,8521	74	184	2,2639	9,4218
35	801	2,9034	8,8755	75	140	2,1458	9,4303
36	809	2,9081	8,8980	76	95	1,9767	9,4386
37	817	2,9122	8,9198	77	48	1,6840	9,4468
38	823	2,9156	8,9409	78	add	add	9,4546
39	829	2,9184	8,9614	79	49	1,6910	9,4624
40	833	2,9205	8,9813	80	100	2,0006	9,4701

	add	add	ſubtract		add	add	ſubtract
81°	152	2,1827	9,4777	121°	3547	3,5499	9,7516
82	206	2,3139	9,4851	122	3675	3,5653	9,7602
83	261	2,4170	9,4924	123	3806	3,5805	9,7690
84	318	2,5021	9,4996	124	3941	3,5956	9,7781
85	376	2,5750	9,5067	125	4078	3,6105	9,7874
86	435	2,6386	9,5138	126	4220	3,6253	9,7971
87	496	2,6951	9,5207	127	4364	3,6399	9,8070
88	558	2,7470	9,5275	128	4513	3,6545	9,8172
89	622	2,7936	9,5342	129	4667	3,6690	9,8278
90	688	2,8373	9,5409	130	4825	3,6835	9,8387
91	754	2,8776	9,5475	131	4989	3,6980	9,8500
92	823	2,9152	9,5540	132	5158	3,7125	9,8618
93	892	2,9506	9,5605	133	5332	3,7269	9,8740
94	963	2,9837	9,5670	134	5512	3,7413	9,8866
95	1036	3,0154	9,5734	135	5698	3,7557	9,8997
96	1110	3,0455	9,5797	136	5890	3,7701	9,9132
97	1186	3,0742	9,5860	137	6088	3,7845	9,9273
98	1264	3,1016	9,5923	138	6297	3,7991	9,9419
99	1343	3,1279	9,5986	139	6513	3,8138	9,9571
100	1423	3,1531	9,6050	140	6739	3,8286	9,9729
101	1505	3,1774	9,6113	141	6974	3,8435	9,9893
102	1588	3,2008	9,6176	142	7219	3,8585	10,0064
103	1673	3,2236	9,6239	143	7475	3,8736	10,0242
104	1761	3,2457	9,6302	144	7743	3,8889	10,0427
105	1850	3,2671	9,6366	145	8024	3,9044	10,0620
106	1941	3,2879	9,6430	146	8316	3,9199	10,0822
107	2033	3,3081	9,6495	147	8630	3,9360	10,1032
108	2127	3,3278	9,6560	148	8962	3,9524	10,1251
109	2223	3,3469	9,6627	149	9312	3,9690	10,1481
110	2321	3,3656	9,6694	150	9681	3,9859	10,1720
111	2421	3,3840	9,6763	151	10072	4,0031	10,1970
112	2523	3,4019	9,6831	152	10491	4,0208	10,2231
113	2627	3,4195	9,6901	153	10940	4,0390	10,2504
114	2734	3,4368	9,6972	154	11416	4,0575	10,2792
115	2843	3,4538	9,7044	155	11929	4,0766	10,3094
116	2954	3,4704	9,7119	156	12483	4,0963	10,3412
117	3068	3,4868	9,7195	157	13086	4,1168	10,3746
118	3184	3,5030	9,7272	158	13740	4,1380	10,4098
119	3302	3,5188	9,7352	159	14455	4,1600	10,4469
120	3422	3,5345	9,7433	160	15223	4,1825	10,4861

An Attempt to ſhew the Advantage ariſing by Taking the Mean of a Number of Obſervations, in practical Aſtronomy.

THOUGH the method practiſed by *Aſtronomers*, in order to diminiſh the errors ariſing from the imperfection of inſtruments and of the organs of ſenſe, by taking the mean of ſeveral obſervations, is of very great utility, and almoſt univerſally followed, yet has it not, that I know of, been hitherto ſubjected to any kind of demonſtration.

In this Eſſay, ſome light is attempted to be thrown on the ſubject, from mathematical principles: in order to the application of which, it ſeemed neceſſary to lay down the following ſuppoſitions.

1. That there is nothing in the conſtruction, or poſition of the inſtrument whereby the errors are conſtantly made to tend the ſame way, but that the reſpective chances for their happening in exceſs, and in defect, are either accurately, or nearly, the ſame.

2. That there are certain aſſignable limits between which all theſe errors may be ſuppoſed to fall; which limits depend on the goodneſs of the inſtrument and the ſkill of the obſerver.

Theſe particulars being premiſed, I ſhall deliver what I have to offer on the ſubject, in the following Propoſitions.

PROPOSITION I.

Suppoſing that the ſeveral chances for the different errors that any ſingle obſervation can admit of, are expreſſed by the terms of the ſeries $r^{-v} \ldots\ldots r^{-3}, r^{-2}, r^{-1}, r^{0}, r^{1}, r^{2}, r^{3} \ldots\ldots r^{v}$; *where the exponents denote the quantities and qualities of the reſpective errors, and the terms themſelves, the reſpective chances for their happening; it is propoſed to determine the probability, or* *odds, that the error, by taking the mean of a given number (n) of obſervations,* exceeds not a given quantity $\left(\frac{m}{n}\right)$.

It is evident from the laws of chance, that, if the given ſeries $r^{-v} \ldots\ldots + r^{-3} + r^{-2} + r^{-1} + r^{0} + r^{1} + r^{2} + r^{3} \ldots\ldots + r^{v}$, expreſſing all the chances in one obſervation, be raiſed to the nth power, the terms of the ſeries thence ariſing will truly exhibit all the different chances in all the propoſed (n) obſervations. But in order to raiſe this power, with the greateſt facility, our given ſeries may be reduced to $r^{-v} \times \frac{1 - r^{2v+1}}{1 - r}$ (by the known rule for ſumming up the terms of a geometrical progreſſion); whereof the nth power (making $w = 2v + 1$) will be $r^{-nv} \times \overline{1 - r^{w}}|^{n} \times \overline{1 - r}|^{-n}$; which expanded, becomes $r^{-nv} - nr^{w-nv} + \frac{n}{1} \cdot \frac{n-1}{2} r^{2w-nv} - \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} r^{3w-nv} + \&c.$ × into $1 + nr + \frac{n}{1} \cdot \frac{n+1}{2} r^{2} + \frac{n}{1} \cdot \frac{n+1}{2} \cdot \frac{n+2}{3} r^{3} + \frac{n}{1} \cdot \frac{n+1}{2} \cdot \frac{n+2}{3} \cdot \frac{n+3}{4} r^{4} + \&c.$

Now, to find from hence the ſum of all the chances, whereby the exceſs of the poſitive errors above the negative ones, can amount to a given number m preciſely, it will be ſufficient (inſtead of multiplying the former ſeries by the whole of the latter) to multiply by ſuch terms of the latter only, as are neceſſary to the production of the given exponent m, in queſtion.

Thus the firſt term (r^{-nv}) of the former ſeries, is to be multiplied by that term of the ſecond whoſe exponent is $nv + m$, in order that the power of r, in the product, may be r^{m}: but it is plain, from the law of the ſeries, that the coefficient of this term (putting $nv + m = q$) will be $\frac{n}{1} \cdot \frac{n+1}{2} \cdot \frac{n+2}{3} \cdot \frac{n+3}{4} (q)$, q being the number of factors; and, conſequently, that the product under conſideration will be $\frac{n}{1} \cdot \frac{n+1}{2} \cdot \frac{n+2}{3} \cdot \frac{n+3}{4} (q) \times r^{m}$.

Again, the ſecond term of the former ſeries being $-nr^{w-nv}$, the exponent of the correſponding term of the latter muſt therefore be $-w + nv + m \; (= q - w)$, and the term itſelf, $\frac{n}{1}.\frac{n+1}{2}.\frac{n+2}{3}(q-w)\times r^{q-w}$; which, drawn into $-nr^{w-nv}$, gives $\frac{n}{1}.\frac{n+1}{2}.\frac{n+2}{3}(q-w)\times nr^{m}$ for the ſecond term required.

In like manner the third term of the product whoſe exponent is m, will be found, $\frac{n}{1}.\frac{n+1}{2}.\frac{n+2}{3}(q-2w)\times\frac{n}{1}.\frac{n-1}{2}r^{m}$. And the ſum of all the terms, having the ſame, given exponent, will conſequently be

$$\begin{array}{l} +\frac{n}{1}.\frac{n+1}{2}.\frac{n+2}{3}.\frac{n+3}{4}(q)\times r^{m} \\ -\frac{n}{1}.\frac{n+1}{2}.\frac{n+2}{3}.\frac{n+3}{4}(q-w)\times nr^{m} \\ +\frac{n}{1}.\frac{n+1}{2}.\frac{n+2}{3}.\frac{n+3}{4}(q-2w)\times\frac{n}{1}.\frac{n-1}{2}r^{m} \\ -\frac{n}{1}.\frac{n+1}{2}.\frac{n+2}{3}.\frac{n+3}{4}(q-3w)\times\frac{n}{1}.\frac{n-1}{2}.\frac{n-2}{3}r^{m} \\ \quad\&c. \qquad\qquad\qquad\qquad\qquad\qquad \&c. \end{array}$$

From which general expreſſion, by expounding m by 0, $+1$, -1, $+2$, -2 *&c.* ſucceſſively, the ſum of the ſeveral chances whereby the difference of the poſitive and negative errors can fall within the propoſed limits $(+m,\ -m)$ will be found: which divided by the total of all the chances, or $r^{-nw}\times\overline{1-r^{w}}|^{n}\times\overline{1-r}|^{-n}$, will be the true meaſure of the probability ſought. From whence the advantage, by taking the mean of ſeveral obſervations, might be made to appear: but this will be ſhewn more properly in the next Propoſition; which is better adapted, and to which this is premiſed, as a *Lemma*.

REMARK.

If r be taken $=1$, or the chances for the poſitive, and the negative errors be ſuppoſed accurately the ſame; then our expreſſion, by expunging the powers of r, will be the very ſame with that ſhewing the chances for throwing $n+q$ points, preciſely, with n dice, each die having as many faces (w) as the reſult of any one ſingle obſervation can come out different ways. Which may be made to appear, independent of any kind of calculation, from the bare conſideration, that the chances for throwing, preciſely, the number m, with n dice, whereof the faces, of each, are numbered $-v \ldots\ldots -3$, -2, -1, -0, $+1$, $+2$, $+3 \ldots\ldots +v$, muſt be the very ſame as the chances whereby the poſitive errors can exceed the negative ones by that preciſe number: but the former are, evidently, the ſame as the chances for throwing preciſely the number $\overline{v+1} \times n + m$ (or $n+q$) with the ſame n dice, when they are numbered in the common way, with the terms of the natural progreſſion 1, 2, 3, 4, 5, and ſo on; becauſe the number upon each face being, *here*, increaſed by $v+1$, the whole increaſe upon all the (n) faces will be expreſſed by $\overline{v+1} \times n$; ſo that there will be, *now*, the very ſame chances for the number $\overline{v+1} \times n + m$, as there was *before* for the number m; ſince the chances for throwing any faces aſſigned will continue the ſame, however thoſe faces are numbered.

PROPOSITION II.

Suppoſing the reſpective chances for the different errors, which any ſingle obſervation can admit of, to be expreſſed by the terms of the ſeries $r^{-v} + 2r^{1-v} + 3r^{2-v} \ldots\ldots + \overline{v+1} . r^{0} \ldots\ldots 3r^{v-2} + 2r^{v-1} + r^{v}$ *(whereof the coefficients, from the middle one* ($v+1$) *decreaſe both ways, according to the terms of an arithmetical progreſſion); it is propoſed to find the probability, or odds, that the error, by taking the mean of a given number* (t) *of obſervations, exceeds not a given quantity* $\left(\frac{m}{t}\right)$.

Following the method laid down in the preceding propoſition, the ſum, or value of the ſeries here propoſed will appear to be $\frac{r^{-v} \times \overline{1-r^{v+1}}|^{2}}{\overline{1-r}|^{2}}$ (being the ſame with the ſquare of the geometrical progreſſion $r^{-\frac{1}{2}v} \times \overline{1+r+r^{2}+r^{3} \ldots\ldots + r^{v}}$). And the power thereof whoſe exponent is t (by making $n = 2t$, and $w = v+1$) will therefore be $r^{-tv} \times \overline{1-r^{w}}|^{n} \times \overline{1-r}|^{-n}$ $= r^{-tv} - nr^{w-tv} + \frac{n}{1} . \frac{n-1}{2} r^{2w-tv} -$ &c. into $1 + nr + \frac{n}{1}$.

$\frac{n+1}{2}r^2 + \frac{n}{1}.\frac{n+1}{2}.\frac{n+2}{3}r^3 +$ &c. Which two ſeries's being the ſame with thoſe in the preceding Problem (excepting only, that the exponents in the former of them are expreſſed in terms of t, inſtead of n) it is plain, that, if q be here put $= tv + m$ (inſtead of $nv + m$) the concluſion there brought out will anſwer equally here; and conſequently that the ſum of all the chances, whereby the exceſs of the poſitive errors, above the negative ones, can amount to the given number m, preciſely, will here, alſo, be truly defined by

$$\begin{aligned}
&+\frac{n}{1}.\frac{n+1}{2}.\frac{n+2}{3}(q) \times r^m \\
&-\frac{n}{1}.\frac{n+1}{2}.\frac{n+2}{3}(q-w) \times nr^m \\
&+\frac{n}{1}.\frac{n+1}{2}.\frac{n+2}{3}(q-2w) \times \frac{n}{1}.\frac{n-1}{2}r^m \\
&-\frac{n}{1}.\frac{n+1}{2}.\frac{n+2}{3}(q-3w) \times \frac{n}{1}.\frac{n-1}{2}.\frac{n-2}{3}r^m \\
&\qquad\quad \&c. \qquad\qquad\qquad\qquad\qquad \&c.
\end{aligned}$$

But this general expreſſion, as ſeveral of the factors in the numerators and denominators mutually deſtroy each other, may be transformed to another more commodious.

Thus the quantity $\frac{n}{1}.\frac{n+1}{2}.\frac{n+2}{3}(q)$, in the firſt line, by breaking the numerator and denominator, each into two parts, will become

$$\frac{n \,.\, n+1 \,.\, n+2 \,.\, n+3 \ldots\ldots q \,.\, q+1 \,.\, q+2 \,.\, q+3 \ldots\ldots q+n-1}{1 \,.\, 2 \,.\, 3 \,.\, 4 \ldots\ldots n \,.\, n+1 \,.\, n+2 \,.\, n+3 \ldots\ldots q};$$

which, by equal diviſion, is reduced to

$$\frac{q+1 \,.\, q+2 \,.\, q+3 \ldots\ldots\ldots q+n-1}{1 \,.\, 2 \,.\, 3 \ldots\ldots\ldots n-1} =$$

$$\frac{\overline{q+n-1} \,.\, \overline{q+n-2} \,.\, \overline{q+n-3} \ldots q+1}{1 \,.\, 2 \,.\, 3 \ldots\ldots n-1} = \frac{p-1}{1}.\frac{p-2}{2}.\frac{p-3}{3}(n-1);$$

ſuppoſing $p = q + n = tv + m + n$.

In the very ſame manner, by making $q' = q - w$, and $p' = q' + n \; (= p - w)$ it appears that $\frac{n}{1}.\frac{n+1}{2}.\frac{n+2}{3}(q-w)$ $=$

$=\frac{p'-1}{1}.\frac{p'-2}{2}.\frac{p'-3}{3}(n-1)$ *&c.* Conſequently our whole given expreſſion (making $p''=p-2w$, $p'''=p-3w$, *&c.*) will be transformed to

$$
\begin{aligned}
&+\frac{p-1}{1}.\frac{p-2}{2}.\frac{p-3}{3}(n-1)\times r^m\\
&-\frac{p'-1}{1}.\frac{p'-2}{2}.\frac{p'-3}{3}(n-1)\times nr^m\\
&+\frac{p''-1}{1}.\frac{p''-2}{2}.\frac{p''-3}{3}(n-1)\times \frac{n}{1}.\frac{n-1}{2}r^m\\
&-\frac{p'''-1}{1}.\frac{p'''-2}{2}.\frac{p'''-3}{3}(n-1)\times \frac{n}{1}.\frac{n-1}{2}.\frac{n-2}{3}r^m\\
&\quad\&c. \qquad\qquad\qquad\qquad\qquad \&c.
\end{aligned}
$$

Which expreſſion is to be continued till ſome of the factors become nothing, or negative; and which, when $r=1$, is the very ſame with that exhibiting the number of chances for p points, preciſely, on n dice, having each w faces.

And, in this caſe, where the chances for the errors in exceſs and in defect are the ſame, the ſolution is the moſt ſimple it can be; ſince, from the chances above determined, anſwering to the number p preciſely, the ſum of the chances for all the inferior numbers to p, may be readily obtained, being given (from the method of increments) equal to $\frac{p-1}{1}.\frac{p-2}{2}.\frac{p-3}{3}(n)$ $-\frac{p'-1}{1}.\frac{p'-2}{2}.\frac{p'-3}{3}(n)\times n+\frac{p''-1}{1}.\frac{p''-2}{2}.\frac{p''-3}{3}(n)\times\frac{n}{1}.\frac{n-1}{2}$ $-\frac{p'''-1}{1}.\frac{p'''-2}{2}.\frac{p'''-3}{3}(n)\times\frac{n}{1}.\frac{n-1}{2}.\frac{n-2}{3}+$ *&c.* The difference between which and half (w^n) the ſum of all the chances (which difference I ſhall denote by D) will conſequently be the true number of the chances whereby the errors in exceſs (or in defect) can fall within the given limit (m): ſo that $\frac{D}{\frac{1}{2}w^n}$ will be the true meaſure of the required probability, that the error, by taking the mean of t obſervations, exceeds not the quantity $\frac{m}{t}$ propoſed.

But now, to illuſtrate this by an example, from whence the utility of the method in practice may clearly appear, it will be neceſſary, in the firſt place, to aſſign ſome number for v, expreſſing the limits of the errors to which any obſervation is ſubject. Theſe limits indeed (as has been before obſerved) depend on the goodneſs of the inſtrument, and the ſkill of the obſerver: but I ſhall here ſuppoſe, that every obſervation may be relied on, to five ſeconds; and that the chances for the ſeveral errors $-5''$, $-4''$, $-3''$, $-2''$, $-1''$, $0''$, $+1''$, $+2''$, $+3''$, $+4''$, $+5''$, included within the limits thus aſſigned, are reſpectively proportional to the terms of the ſeries 1, 2, 3, 4, 5, 6, 5, 4, 3, 2, 1. Which ſeries is much better adapted, than if all the terms were to be equal; ſince it is highly reaſonable to ſuppoſe, that the chances for the reſpective errors decreaſe, in proportion as the errors themſelves increaſe.

Theſe particulars being premiſed, let it be now required to find what the probability, or chance for an error of 1, 2, 3, 4, or 5 ſeconds will be, when, inſtead of relying on one, the mean of ſix obſervations is taken.

Here, v being $= 5$, and $t = 6$, we ſhall have $n\ (= 2t) = 12$, $w\ (= v + 1) = 6$, and $p\ (= tv + n + m) = 42 + m$: but the value of m, if we firſt ſeek the chances whereby the error exceeds not one ſecond, will be had from the equation $\frac{m}{t} = \pm 1$; where either ſign may be uſed (the chances being the ſame) but the negative one is the moſt commodious: from whence we have $m\ (= -t) = -6$; and therefore $p = 36$, $p' = 30$, $p'' = 24$, &c.

Which values being ſubſtituted in the general expreſſion above determined, it will become $\frac{35}{1}.\frac{34}{2}.\frac{33}{3}(12) - \frac{29}{1}.\frac{28}{2}.\frac{27}{3}(12) \times 12 + \frac{23}{1}.\frac{22}{2}.\frac{21}{3}(12) \times 66 - \frac{17}{1}.\frac{16}{2}.\frac{15}{3}(12) \times 220 = 299576368$: and this ſubtracted from $1088391168\ (= \frac{1}{2} \times 6^{12})$, leaves 78881480, for the value of D correſponding: therefore the required probability that the error, by taking the mean of ſix obſervations, exceeds not a ſingle *ſecond*, will be truly meaſured by the fraction $\frac{788814800}{1088391168}$; and conſequently the odds will be as 788814800 to 299576368, or nearly as $2\frac{1}{2}$ to 1. But the odds, or proportion, when one ſingle obſervation is taken, is only as 16 to 20, or as $\frac{8}{10}$ to 1.

To determine, now, the probability that the reſult comes within two ſeconds of the truth, let $\frac{m}{t}$ be made $= -2$; ſo ſhall $m\ (= -2t) = -12$: therefore $p = 30$, $p' = 24$, $p'' = 18$, &c. and our general expreſſion will here come out $= 36079407$; whence $D = 1052311761$. Conſequently $\frac{1052311761}{1088391168}$ will be the true meaſure of the probability ſought: and the odds, or proportion of the chances, will therefore be that of 1052311761 to 36079407, or as 29 to 1, nearly. But the proportion, or odds, when a ſingle obſervation is taken, is only as 2 to 1: ſo that the chance for an error exceeding two ſeconds, is not $\frac{1}{10}$th part ſo great, from the mean of ſix, as from one ſingle obſervation. And it will be found in the ſame manner, that the chance for an error exceeding three ſeconds is not *here* $\frac{1}{1000}$ part ſo great as it will be from one obſervation only. Upon the whole of which it appears, that, the taking of the mean of a number of obſervations, greatly diminiſhes the chances for all the ſmaller errors, and cuts off almoſt all poſſibility of any large ones: which laſt conſideration alone is ſufficient to recommend the uſe of the method, not only to *Aſtronomers*, but to all *Others* concerned in making experiments, or obſervations of any kind, which will allow of being repeated under the ſame circumſtances.

In the preceding calculations, the different errors to which any obſervation is ſuppoſed ſubject, are reſtrained to whole quantities, or a certain, preciſe, number of ſeconds; it being impoſſible, from the moſt exact inſtruments, to take off the quantity of an angle to a *geometrical exactneſs*. But I ſhall now ſhew how the chances may be computed, when the error admits of any value whatever, whole or broken, within the propoſed limits, or when the reſult of each obſervation is ſuppoſed to be *accurately* known.

Let

Fig. 20. Let, then, the line AB repreſent the whole extent of the given interval, within which all the obſervations are ſuppoſed to fall; and conceive the ſame to be divided into an exceeding great number of very ſmall, equal particles, by perpendiculars terminating in the ſides AD, BD of an iſoſceles triangle ABD, formed upon the baſe AB: and let the probability or chance whereby the reſult of any obſervation tends to fall within any of theſe very ſmall intervals N*n*, be proportional to the correſponding area NM*mn*, or to the perpendicular NM; then, ſince theſe chances (or areas) reckoning from the extremes A and B, increaſe according to the terms of the arithmetical progreſſion 1, 2, 3, 4, *&c.* it is evident that the caſe is here the ſame with that in the latter part of Prop. II.; only, as the number v (expreſſing the particles in AC or BC) is indefinitely great, all (finite) quantities joined to v, or its multiples, with the ſigns of addition or ſubtraction, will here vaniſh, as being nothing in compariſon of v. By which means the general expreſſion $\left(\frac{p-1}{1}\cdot\frac{p-2}{2}\cdot\frac{p-3}{3}(n) - \frac{p'-1}{1}\cdot\frac{p'-2}{2}\cdot\frac{p'-3}{3}(n)\times n + \frac{p''-1}{1}\cdot\frac{p''-2}{2}\cdot\frac{p''-3}{3}(n)\times n\cdot\frac{n-1}{2}, \text{\&c.}\right)$ *there* determined, will *here* become $\frac{p}{1}\cdot\frac{p}{2}\cdot\frac{p}{3}(n) - \frac{p'}{1}\cdot\frac{p'}{2}\cdot\frac{p'}{3}(n)\times n, \text{\&c.} = \frac{1}{1.2.3.4(n)} \times \overline{p^n - np'^n + n\cdot\frac{n-1}{2}p''^n - n\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}p'''^n, \text{\&c.}}$ (wherein $p = tv \mp m$, $p' = p - v$, $p'' = p - 2v$, $p'' = p - 3v$, *&c.*) and therefore, the value of D in the preſent caſe, being $\frac{1}{2}v^n - \frac{1}{1.2.3(n)} \times \overline{p^n - n\cdot\overline{p-v}\rceil^n + n\cdot\frac{n-1}{2}\cdot\overline{p-2v}\rceil^n}$, *&c.* it is evident that the probability $\left(\frac{D}{\frac{1}{2}v^n}\right)$ of the error's not exceeding the quantity $\frac{m}{t}$ (in taking the mean of t obſervations) will be truly defined by

$$1 - \frac{2}{1.2.3(n)} \times \overline{\overline{\frac{p}{v}}\Big\rceil^n - n\times\overline{\frac{p}{v} - 1}\Big\rceil^n + \frac{n}{1}\cdot\frac{n-1}{2}\times\overline{\frac{p}{v} - 2}\Big\rceil^n, \text{\&c.}}$$

which may be repreſented by the curvilineal area CNFE, correſponding to the given value or abſciſſa CN $(= \frac{m}{t})$. Now, though the numbers v, p, and m are, all of them, here ſuppoſed to be indefinitely great, yet they may be exterminated, and the value of the expreſſion determined, from their known relation to each other. For if the given ratio of $\frac{m}{t}$ to v, or of CN to CA, be expreſſed by that of x to 1, or, which is the ſame, if the error in queſtion be ſuppoſed the x part of the greateſt error; then, m being $= tvx$, p $(= tv \mp m)$ will be $= tv \mp tvx$, and therefore $\frac{p}{v} = t \times \overline{1 \mp x}$; which let be denoted by y: then, by ſubſtitution, our laſt general expreſſion will become

$$1 - \frac{2}{1.2.3(n)} \times \begin{cases} y^n - n.\overline{y-1}|^n + \frac{n}{1}.\frac{n-1}{2}.\overline{y-2}|^n - \\ \frac{n}{1}.\frac{n-1}{2}.\frac{n-2}{3}.\overline{y-3}|^n, \text{ \&c.} \end{cases}$$

which ſeries is to be continued till the quantities y, $y - 1$, $y - 2$, &c. become negative.

As an example of what is above delivered, let it be now required to find the probability, or odds that the error, by taking the mean of ſix obſervations, exceeds not a ſingle ſecond; ſuppoſing (as in the former example) that the greateſt error, that any obſervation can admit of, is limited to five ſeconds.

Here t being $= 6$, n $(= 2t) = 12$, and $x = \frac{1}{5}$, we have y $(= t \times \overline{1 - x}) = 4{,}8$; and therefore the meaſure of the probability ſought will be equal to $1 - \frac{2}{1.2.3(12)} \times$ $\overline{\overline{4{,}8}|^{12} - 12 \times \overline{3{,}8}|^{12} + 66 \times \overline{2{,}8}|^{12} - 220 \times \overline{1{,}8}|^{12} + 495 \times \overline{0{,}8}|^{12}}$ $= 0{,}7668$, nearly: ſo that the odds, that the error exceeds not a ſingle ſecond, will be as 0,7668 to 0,2332; which is more than three to one. But the proportion, when one ſingle obſervation is relied on, is only as 36 to 64, or as 9 to 16. In the ſame manner, taking $x = \frac{2}{5}$, it will be found, that the odds, of the error's not exceeding two ſeconds, when the mean of ſix obſervations is taken, will be as 0,985 to 0,015, nearly, or as $65\frac{2}{3}$ to 1; whereas the odds on one ſingle obſervation, is only as 64 to 36, or as $1\frac{7}{9}$ to 1: ſo that the chance for an error of two ſeconds is not $\frac{1}{20}$th part ſo great, from the mean of ſix, as from one ſingle obſervation. And it will farther appear, by making $x = \frac{3}{5}$, that the probability of an error of three ſeconds, here, is not $\frac{1}{1400}$th part ſo great as from one ſingle obſervation: ſo that in this, as well as in the former hypotheſis, almoſt all poſſibility of any large error is cut off. And the caſe will be found the ſame, whatever hypotheſis is aſſumed to expreſs the chances for the errors to which any ſingle obſervation is ſubject.

From the ſame general expreſſion by which the foregoing proportions are derived, it will be eaſy to determine the odds, that the mean of a given number of obſervations is nearer to the truth than one ſingle obſervation, taken indifferently. For, if z be put $(= 1 - x) = \frac{y}{t}$, and $s = \frac{1}{t}$, then, y being $= tz$, the quantity $\frac{2}{1.2.3(n)} \times \overline{y^n - n.\overline{y-1}|^n + n.\frac{n-1}{2}.\overline{y-2}|^n, \&c.}$ (expreſſing the probability that the reſult falls within the diſtance z of the greateſt limit) will here, by ſubſtitution, become $\frac{2t^n}{1.2.3(n)} \times \overline{z^n - n \times \overline{z-s}|^n + n.\frac{n-1}{2} \times \overline{z-2s}|^n, \&c.}$ which, in caſe of one ſingle obſervation (when $t = 1$, and $n = 2$) is barely z^2, and its fluxion $2z\dot{z}$: therefore, if we now multiply by $2z\dot{z}$, the product $\frac{4t^n}{1.2.3(n)} \times \overline{z^{n+1}\dot{z} - n.\overline{z-s}|^n.z\dot{z} + n.\frac{n-1}{2}.\overline{z-2s}|^n.z\dot{z}, \&c.}$ will give the fluxion of the probability that the reſult of t obſervations is farther from the truth, or nearer to the limits, than one ſingle obſervation taken indifferently. And conſequently the fluent thereof, which is $\frac{4t^n}{1.2.3(n)}$ into $\frac{z^{n+2}}{n+2}$

—

$$-\frac{n}{1}\times\overline{\frac{s.\overline{z-s}|^{n+1}}{n+1}+\frac{\overline{z-s}|^{n+2}}{n+2}}+\frac{n}{1}.\frac{n-1}{2}\times\overline{\frac{2s.\overline{z-2s}|^{n+1}}{n+1}+\frac{\overline{z-2s}|^{n+2}}{n+2}}$$

&c. will, when $z=1$, be the true meaſure of the probability itſelf. Which, in the caſe above propoſed, where $t=6$, and $n=12$, will be found $=0{,}245$, and, conſequently, the odds that the mean of ſix, is nearer to the truth than one ſingle obſervation, as 755 to 245, or as 151 to 49.

A
DETERMINATION
OF
Certain FLUENTS, and the RESOLUTION of ſome very uſeful EQUATIONS in the higher Orders of Fluxions; by means of the Meaſures of Angles and Ratios, and the Right-ſines and Verſed-ſines of circular Arcs.

IN order to treat the matter here propoſed with due perſpicuity, it will be neceſſary, previous thereto, to give a demonſtration of the two ſubſequent *Lemmas.*

LEMMA I.

The double of the rectangle contained under the co-ſines of any two arcs, ſuppoſing the radius to be unity, is equal to the ſum of the co-ſines of the ſum, and difference of thoſe arcs.

Fig. 21. For, let AB and BD (= BE) be the two arcs, and CG and Cn their reſpective co-ſines; likewiſe let CH be the co-ſine of their ſum AE, and CF that of their difference AD; making nm parallel to BG. Then, Dn being = En, it follows that Fm = Hm; and conſequently that $2\text{C}m = \text{CH} + \text{CF}$: but, by ſimilar triangles, $\text{CB} : \text{CG} :: \text{C}n : \text{C}m$; whence $\text{CG} \times 2\text{C}n = \text{CB} \times 2\text{C}m = \text{CB} \times \overline{\text{CH} + \text{CF}}$. *Q. E. D.*

LEMMA II.

If A *be any arch of a circle whoſe radius is unity, and n any whole poſitive number; then will*

$$\overline{\textit{co-ſin.}\ \text{A}}\big|^{n} = \frac{1}{2}\Big|^{n-1} \textit{ into co-ſin.}\ n\text{A} + n\ \textit{co-ſin.}\ \overline{n-2}\,.\,\text{A} + n\,.\,\frac{n-1}{2}\ \textit{co-ſin.}\ \overline{n-4}\,.\,\text{A} + n\,.\,\frac{n-1}{2}\,.\,\frac{n-2}{3}\ \textit{co-ſin.}\ \overline{n-6}\,.\,\text{A} + \textit{\&c. continued}$$

to $\frac{n+1}{2}$, *or* $\frac{1}{2}n+1$ *terms, according as n is an odd, or even number; in the latter of which caſes the half, only, of the laſt term is to be taken.*

For, by the preceding *Lemma*, $2\,\overline{\text{coſ. A}}\rvert^{2} = \text{coſ.}\ \overline{A+A} + \text{coſ.}\ \overline{A-A} = \text{coſ. } 2A + 1$; which equation multiplied by 2 coſ. A, gives $2^{2}\times\overline{\text{coſ. A}}\rvert^{3} = \text{coſ. } 2A \times 2\ \text{coſ. } A + 2\ \text{coſ. } A = \text{coſ. } 3A + \begin{matrix}\text{coſ. A}\\ 2\,\text{coſ. A}\end{matrix}$ *(by Lem. I.)* $= \text{coſ. } 3A + 3\ \text{coſ. } A$.

Multiply, again, by 2 coſ. A; ſo ſhall $2^{3}\times\overline{\text{coſ. A}}\rvert^{4} = \text{coſ. } 3A \times 2\ \text{coſ. } A + 3\ \text{coſ. } A \times 2\ \text{coſ. } A = \text{coſ. } 4A + \begin{matrix}\text{coſ. } 2A\\ \text{coſ. } 2A\end{matrix} + 3\ \text{coſ. } 0$ *(by Lemma I.)* $= \text{coſ. } 4A + 4\ \text{coſ. } 2A + 3$. In the ſame manner we have,

$2^{4}\times\overline{\text{coſ. A}}\rvert^{5} = \text{coſ. } 5A + 5\,\text{coſ. } 3A + 10\,\text{coſ. } A$,
$2^{5}\times\overline{\text{coſ. A}}\rvert^{6} = \text{coſ. } 6A + 6\,\text{coſ. } 4A + 15\,\text{coſ. } 2A + 10$,
$2^{6}\times\overline{\text{coſ. A}}\rvert^{7} = \text{coſ. } 7A + 7\,\text{coſ. } 5A + 21\,\text{coſ. } 3A + 35\,\text{coſ. } A$,.
&c. *&c.*

where the law of continuation is manifeſt; the numeral coefficients being the ſame, and generated in the very ſame manner, with thoſe of a binomial raiſed to the 2d, 3d, 4th, 5th, *&c.* powers, ſucceſſively; except in the laſt term, when the exponent *n* is even, in which caſe one half only of the correſpondent (or middle) term of the involved binomial is concerned. Hence the propoſition is manifeſt.

The ſame otherwiſe.

If the co-ſine of A be denoted by x, it is well known that $\dot{A} = \frac{-\dot{x}}{\sqrt{1-xx}}$. Multiply the whole equation by $\sqrt{-1}$, ſo ſhall $\dot{A}\sqrt{-1} = \frac{-\dot{x}\sqrt{-1}}{\sqrt{1-xx}} = \frac{-\dot{x}\times-1}{\sqrt{1-xx}\times\sqrt{-1}} = \frac{\dot{x}}{\sqrt{-1+xx}}$: whence, by taking the fluent, we have $A\sqrt{-1} = \text{hyp. log. } \overline{x+\sqrt{xx-1}}$. Let N be the number whoſe hyperbolical logarithm is 1; then, ſince hyp. log. $N^{A\sqrt{-1}}$ $(=A\sqrt{-1})$ = hyp. log. $x+\sqrt{xx-1}$, it is evident that $N^{A\sqrt{-1}}=x+\sqrt{xx-1}$, or $M^{A}=x+\sqrt{xx-1}$ (by making $M=N^{\sqrt{-1}}$). From which equation x (the co-ſine of A) is found $=\frac{M^{A}+M^{-A}}{2}$ (and from thence $(\sqrt{1-xx})$ the ſine of $A=\sqrt{-1}\times\frac{M^{A}-M^{-A}}{2}$; but this laſt by the bye).

Now ſeeing that $2\text{ coſ. }A=M^{A}+M^{-A}$, we therefore have $\overline{2\text{ coſ. }A}|^{n}=\overline{M^{A}+M^{-A}}|^{n}=\overline{M^{nA}+M^{-nA}}+n\times\overline{M^{\overline{n-2}.A}+M^{\overline{-n+2}.A}}+n.\frac{n-1}{2}\times\overline{M^{\overline{n-4}.A}+M^{\overline{-n+4}.A}}+\&c.$ by expanding $\overline{M^{A}+M^{-A}}|^{n}$ and uniting, in pairs, the correſpondent terms (*viz.* the firſt and laſt, the ſecond and laſt but one, and ſo on). But $M^{nA}+M^{-nA}$, the firſt of theſe pairs, is the double of the co-ſine of nA; for the very ſame reaſon that $M^{A}+M^{-A}$ was found to expreſs the double of the co-ſine of A. And thus, $A^{\overline{n-2}.A}+A^{\overline{-n+2}.A}$ will appear to expreſs the double of the co-ſine of $\overline{n-2}.A$, &c. And our equation will therefore be reduced to $2^{n}\times\overline{\text{coſ. }A}|^{n}=2\text{ coſ. }nA+2n\text{ coſ. }\overline{n-2}.A+2n.\frac{n-1}{2}\text{ coſ. }\overline{n-4}.A+\&c.$ or to $\overline{\text{coſ. }A}|^{n}=\overline{\frac{1}{2}}|^{n-1}$

$$\times\left\{\begin{array}{l}\text{coſ. }nA+n\text{ coſ. }\overline{n-2}.A+n.\frac{n-1}{2}\text{ coſ. }\overline{n-4}.A\\ \qquad+n.\frac{n-1}{2}.\frac{n-2}{3}\text{ coſ. }\overline{n-6}.A+\&c.\end{array}\right.$$

where, when the exponent n is even (the number of terms in $\overline{M^{A}+M^{-A}}|^{n}$, expanded, being odd) there will be a middle term (no-ways effected by M or A) which being an abſolute number, muſt be taken ſingly, and conſequently, only the half thereof when the whole ſeries is divided by 2, as is the caſe in the concluſion. *Q. E. D.*

COROL-

COROLLARY.

If Q be taken to repreſent an arch of 90 degrees, and the complement (Q — A) of the arch A be put = B; then, by ſubſtituting ſin. B for coſ. A, and Q — B for A in our general equation, we ſhall have $\overline{\text{ſin. B}}\big|^{n} = \frac{1}{2}\Big|^{n-1}$ into $\text{coſ.}\ \overline{n\text{Q}-n\text{B}} + n\ \text{coſ.}\ \overline{\overline{n-2}.\text{Q}-\overline{n-2}.\text{B}} + n.\frac{n-1}{2}\ \text{coſ.}\ \overline{\overline{n-4}.\text{Q}-\overline{n-4}.\text{B}}$ + &c. being a general expreſſion for any power of the *ſine* of an arch (as the former was of the co-ſine). But this expreſſion may be reduced to a form ſomewhat more commodious, regard being had to the different interpretations of *n*, with reſpect to even, and odd numbers. Thus, if *n* be expounded by any term of the ſeries 4, 8, 12, 16, &c. it is evident that nQ (in the firſt term) will be an *even* multiple of the ſemi-periphery; and that $\overline{n-2}.\text{Q}$ (in the ſecond term) will be an *odd* one, and ſo on, alternately. But it is well known, that ſubtracting, or caſting off any multiple of the ſemi-periphery no-ways affects the value of the ſine, or co-ſine; except, that ſuch value, when the multiple is an odd one, will be changed from poſitive to negative (and *vice verſâ)*. Hence our laſt equation will be reduced to $\overline{\text{ſin. B}}\big|^{n} = \frac{1}{2}\Big|^{n-1}$ into $\text{coſ.} - n\text{B} - n\ \text{coſ.} - \overline{n-2}\ .\ \text{B} + n\ .\frac{n-1}{2}\ \text{coſ.}\ \overline{n-4}\ .\ \text{B}$ — &c. $= \frac{1}{2}\Big|^{n-1}$ into $\text{coſ.}\ n\text{B} - n\ \text{coſ.}\ \overline{n-2}\ .\ \text{B} + n\ .\ \frac{n-1}{2}\ \text{coſ.}\ \overline{n-4}\ .\ \text{B}$ &c.

And, for the ſame reaſons, the equation, when *n* is interpreted by any term of the ſeries 2, 6, 10, 14, &c. will appear to be $\overline{\text{ſin. B}}\big|^{n} = \frac{1}{2}\Big|^{n-1}$ into $- \text{coſ.}\ n\text{B} + n\ \text{coſ.}\ \overline{n-2}\ .\text{B} - n.\frac{n-1}{2}\ \text{coſ.}\ \overline{n-4}\ .\text{B}$ + &c.

But, when *n* is expounded by any of the odd numbers 1, 5, 9, 13, &c. we ſhall then (by rejecting the multiples of the ſemi-periphery, *&c.*) have $\overline{\text{ſin.}\,B}\rvert^{n} = \overline{\tfrac{1}{2}}\Big\rvert^{n-1}$ into $\text{coſ.}\,\overline{Q - nB} - n\,\text{coſ.}\,\overline{Q - \overline{n-2}.B} + n.\frac{n-1}{2}\,\text{coſ.}\,\overline{Q - \overline{n-4}.B} -$ *&c.* $= \overline{\tfrac{1}{2}}\Big\rvert^{n-1}$ into $\text{ſin.}\,nB - n\,\text{ſin.}\,\overline{n-2}.B + n.\frac{n-1}{2}\,\text{ſin.}\,\overline{n-4}.B -$ *&c.*

Laſtly, if n be expounded by any term of the ſeries 3, 7, 11, 15, *&c.* the reſult, or ſeries, will be the ſame as in the preceding caſe, only the ſigns of all the terms muſt be changed to their contrary.

But all theſe different caſes may be otherwiſe, more directly, inveſtigated, by means of the two following Theorems; whereof the Demonſtration is obvious, from that of *Lemma I***.

1°. *The double of the rectangle contained under the ſines of any two arcs, ſuppoſing the radius to be unity, is equal to the difference of the co-ſines of the ſum, and difference of thoſe arcs.*

2°. *And the double of the rectangle under the co-ſine of the one and the ſine of the other, is equal to the difference of the ſines of the ſum, and difference of the ſaid arcs.*

Hence it follows, that $\text{ſin.}\,B \times 2\,\text{ſin.}\,B = -\,\text{coſ.}\,\overline{B+B} + \text{coſ.}\,\overline{B-B}$ *(by Theor. I.)* $= -\,\text{coſ.}\,2B + 1$: whence, multiplying the whole equation by $2\,\text{ſin.}\,B$, we have $2^2 \times \overline{\text{ſin.}\,B}\rvert^{3} = -\,\text{coſ.}\,2B \times 2\,\text{ſin.}\,B + 2\,\text{ſin.}\,B = -\,\text{ſin.}\,3B + \substack{\text{ſin.}\,B \\ 2\,\text{ſin.}\,B}$ *(by Theor. II.)* $= -\,\text{ſin.}\,3B + 3\,\text{ſin.}\,B$. Whence, again, by equal multiplication, $2^3 \times \overline{\text{ſin.}\,B}\rvert^{4} = -\,\text{ſin.}\,3B \times 2\,\text{ſin.}\,B + 3\,\text{ſin.}\,B \times 2\,\text{ſin.}\,B = +\,\text{coſ.}\,4B - \substack{\text{coſ.}\,2B \\ 3\,\text{coſ.}\,2B} + 3\,\text{coſ.}\,0$ *(by Theorem I.)* $= \text{coſ.}\,4B - 4\,\text{coſ.}\,2B + 3$.

In like manner, $2^4 \times \overline{\text{ſin.}\,B}\rvert^{5} = \text{ſin.}\,5B - 5\,\text{ſin.}\,3B + 10\,\text{ſin.}\,B$; and $2^5 \times \overline{\text{ſin.}\,B}\rvert^{6} = -\,\text{coſ.}\,6B + 6\,\text{coſ.}\,4B - 15\,\text{coſ.}\,2B + 10$, *&c.*

Fig. 21. * By ſim. Δ's, BC(1) : BG :: DE (2D*n*) : D*p*(CF — CH) = 2D*n* × BG: And BC (1) : CG :: DE (2D*n*) : E*p* (EH — DF) = 2CG × D*n*.

Whence, *univerſally*, $\overline{\text{ſin. B}}\big|^{n} = \frac{1}{2}\Big]^{n-1} \times$

$$\overline{\pm \text{ ſin. } nB \mp n \text{ ſin. } \overline{n-2}\,.\,B \pm n\,.\,\frac{n-1}{2}\text{ ſin. } \overline{n-4}\,.\,B \mp \text{\&c.}}$$

when *n* is an odd number; and $\overline{\text{ſin. B}}\big|^{n} = \frac{1}{2}\Big]^{n-1} \times$

$$\overline{\pm \text{ coſ. } nB \mp n \text{ coſ. } \overline{n-2}\,.\,B \pm n\,.\,\frac{n-1}{2}\text{ coſ. } \overline{n-4}\,.\,B \mp \text{\&c.}}$$

when *n* is an even number: where the number of terms, in the former caſe, will be $\frac{n+1}{2}$, and in the latter $\frac{1}{2}n + 1$; in which caſe the half, only, of the laſt term is to be taken; and is always poſitive, as well as the laſt term in the former caſe: whereby the ſigns of all the other terms (as they change alternately) will be known.

If $\alpha, \beta, \gamma, \delta \ldots\ldots M$ be aſſumed to expreſs the terms ($1, n, n\,.\,\frac{n-1}{2}, n\,.\,\frac{n-1}{2}\,.\,\frac{n-2}{3}$, *&c.*) of $1+1$ raiſed to the *n*th power (*M* being the middle, or greateſt term) it is evident that the ſecond caſe of our general equation (wherein *n* is even) will ſtand thus, $\overline{\text{ſin. B}}\big|^{n} = \frac{1}{2}\Big]^{n-1}$ into $\pm\, \alpha \text{ coſ. } nA \mp \beta \text{ coſ. } \overline{n-2}\,.\,A \pm \gamma \text{ coſ. } \overline{n-4}\,.\,A \mp \delta \text{ coſ. } \overline{n-6}\,.\,A \,\ldots\ldots + \frac{1}{2}M.$

By the ſame method of proceeding, an expreſſion exhibiting the continual product of the co-ſines, or ſines of any number of unequal arches, may be derived.

For *(by Lemma I.)* $\text{coſ. A} \times 2 \text{ coſ. B} = \text{coſ. } \overline{A+B} + \text{coſ. } \overline{A-B}$; whence $\text{coſ. A} \times 2 \text{ coſ. B} \times 2 \text{ coſ. C} = \text{coſ. } \overline{A+B} \times 2 \text{ coſ. C} + \text{coſ. } \overline{A-B} \times 2 \text{ coſ. C}$ *(by equal multiplication)*

$$= \text{coſ. } \overline{A+B+C} + \begin{array}{l} \text{coſ. } \overline{A+B-C} \\ \text{coſ. } \overline{A-B+C} \\ \text{coſ. } \overline{-A+B+C} \end{array} \text{(\textit{by the Lemma})};$$

whence, again (by equal multiplication *and the Lemma*) we have $\text{coſ. } A \times 2\,\text{coſ. } B \times 2\,\text{coſ. } C \times 2\,\text{coſ. } D = \text{coſ. }$

$$\overline{A+B+C+D} + \begin{array}{l} \text{coſ. } \overline{A+B+C-D} \\ \text{coſ. } \overline{A+B-C+D} \\ \text{coſ. } \overline{A-B+C+D} \\ \text{coſ. } \overline{-A+B+C+D} \end{array} + \begin{array}{l} \text{coſ. } \overline{A+B-C-D} \\ \text{coſ. } \overline{A-B+C-D} \\ \text{coſ. } \overline{-A+B+C-D} \end{array}$$

from which the law of continuation is manifeſt.

PROBLEM I.

To determine the fluent of $\frac{x^n \dot{x}}{\sqrt{1-xx}}$; *n being any odd affirmative number.*

If A be aſſumed to denote the arch of a circle, whoſe ſine is x and radius 1; it is well known that $\dot{A} = \frac{\dot{x}}{\sqrt{1-xx}}$: and, *by the Corol. to Lem. II,* it alſo appears that

$$x^n = \pm \left.\overline{\tfrac{1}{2}}\right|^{n-1} \times \overline{\text{ſin.}\, nA - n\,\text{ſin.}\,\overline{n-2}\,.\,A + n\,.\,\tfrac{n-1}{2}\,\text{ſin.}\,\overline{n-4}\,.\,A\left(\tfrac{n+1}{2}\right)}.$$

Hence we have

$$\frac{x^n \dot{x}}{\sqrt{1-xx}} \left(= x^n \dot{A}\right) = \pm \left.\overline{\tfrac{1}{2}}\right|^{n} \times \overline{\dot{A} \times \text{ſ.}\, nA - n\dot{A} \times \text{ſ.}\,\overline{n-2}\,.\,A + n\,.\,\tfrac{n-1}{2}\,\dot{A} \times \text{ſ.}\,\overline{n-4}\,.\,A\left(\tfrac{n+1}{2}\right)}.$$

But the fluxion of any arch, multiplied by the ſine (the radius being unity) is equal to the fluxion of the verſed-ſine: therefore the fluxions of the verſed-ſines of the arches nA, $\overline{n-2}\,.\,A$, $\overline{n-4}\,.\,A$, &c. will be $n\dot{A} \times \text{ſin. } nA$, $\overline{n-2}\,.\,\dot{A} \times \text{ſin. } \overline{n-2}\,.\,A$, $\overline{n-4}\,.\,\dot{A} \times \text{ſin. } \overline{n-4}\,.\,A$, &c. reſpectively; and conſequently the ſaid verſed-ſines, the true fluents of theſe fluxions: whence it is manifeſt that the true fluent of our whole expreſſion will be $\pm \left.\overline{\tfrac{1}{2}}\right|^{n-1}$ into $\frac{1}{n} \times \text{verſ. ſin. } nA - \frac{n}{n-2} \times \text{verſed-ſin. } \overline{n-2}\,.\,A + \frac{n}{n-4}\,.\,\frac{n-1}{2} \times \text{verſed-ſin. } \overline{n-4}\,.\,A - \frac{n}{n-6}\,.\,\frac{n-1}{2}\,.\,\frac{n-2}{3} \times \text{verſed-ſin. } \overline{n-6}\,.\,A\left(\frac{n+1}{2}\right)$. Wherein, of the ſigns + and — before $\left.\overline{\tfrac{1}{2}}\right|^{n-1}$, the former, or latter obtains, according as $\frac{n+1}{2}$, expreſſing the number of terms, is odd or even.

PROBLEM II.

To find the fluent of $\frac{x^n\dot{x}}{\sqrt{1-xx}}$; *n being any even affirmative number.*

By the preceding Problem $\frac{\dot{x}}{\sqrt{1-xx}}=\dot{A}$; and, *by the Corol. to Lem. II*, $x^n=\pm\left.\overline{\tfrac{1}{2}}\right|^{n-1}\times$

$$\overline{\alpha\ \text{coſ.}\ nA-\beta\times\text{coſ.}\,\overline{n-2}.A+\gamma\times\text{coſ.}\,\overline{n-4}.A\ldots\ldots\pm\tfrac{1}{2}M}.$$

Therefore $\frac{x^n\dot{x}}{\sqrt{1-xx}}\ (=x^n\dot{A})=\pm\left.\overline{\tfrac{1}{2}}\right|^{n-1}\times$

$$\overline{\alpha\dot{A}\times\text{coſ.}\,nA-\beta\dot{A}\times\text{coſ.}\,\overline{n-2}.A+\gamma\dot{A}\times\text{coſ.}\,\overline{n-4}.A\ldots\ldots\pm\tfrac{1}{2}M\dot{A}}.$$

But the fluxion of any arch, multiplied by the co-ſine, is equal to the fluxion of the ſine, drawn into the radius: whence it follows that $n\dot{A}\times\text{coſ.}\,nA$, $\overline{n-2}.\dot{A}\times\text{coſ.}\,\overline{n-2}.A$, $\overline{n-4}.\dot{A}\times\text{coſ.}\,\overline{n-4}.A$, *&c.* are the fluxions of the ſines of the arches nA, $\overline{n-2}.A$, $\overline{n-4}.A$, *&c.* reſpectively; and, conſequently that

$$\pm\left.\overline{\tfrac{1}{2}}\right|^{n-1}\times\left\{\begin{array}{l}\frac{\alpha}{n}\times\text{ſin.}\,nA-\frac{\beta}{n-2}\times\text{ſin.}\,\overline{n-2}.A+\frac{\gamma}{n-4}\times\text{ſin.}\\ \qquad\overline{n-4}.A-\frac{\delta}{n-6}\times\text{ſin.}\,\overline{n-6}.A\ldots\pm\tfrac{1}{2}MA\end{array}\right.$$

will be the true fluent ſought: where $\alpha=1$, $\beta=n$, $\gamma=\beta\times\frac{n-1}{2}$, $\delta=\gamma\times\frac{n-2}{3}$, *&c.* and wherein the ſign + or —, before $\left.\overline{\tfrac{1}{2}}\right|^{n-1}$, obtains according as $\tfrac{1}{2}n+1$, expreſſing the number of terms, is odd, or even.

COROLLARY I.

Since the value (M) of the middle term of the binomial $\overline{1+1}|^{n}$, expanded in a ſeries, is known to be $\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}$ $\ldots\ldots\frac{\frac{1}{2}n+1}{\frac{1}{2}n}$, by the law of the ſeries, it is evident that the term next adjacent to it (on either ſide) will be expreſſed by $M\times\frac{\frac{1}{2}n}{\frac{1}{2}n+1}$, or by $M\times\frac{m}{m+1}$; making $m=\frac{1}{2}n$. And, in the ſame manner, it will appear that the next term to this laſt will be expreſſed by $M\times\frac{m}{m+1}\times\frac{m-1}{m+2}$; and the next to that, by $M\times\frac{m}{m+1}\times\frac{m-1}{m+2}\times\frac{m-2}{m+3}$; and ſo on. Therefore, by ſubſtituting theſe values above, and inverting the order of the terms, the general fluent, there given, will here be transformed to

$$\frac{M}{2^{n}}\times\begin{cases}A-\frac{m}{m+1}\times\text{ſin. }2A+\frac{m}{m+1}\times\frac{m-1}{m+2}\times\frac{1}{2}\text{ſin. }4A-\frac{m}{m+1}\times\\ \frac{m-1}{m+2}\times\frac{m-2}{m+3}\times\frac{1}{3}\text{ſ. }6A+\frac{m}{m+1}\times\frac{m-1}{m+2}\times\frac{m-2}{m+3}\times\frac{m-3}{m+4}\times\frac{1}{4}\text{ſ. }8A\end{cases}$$

&c. where the ſeries is to be continued till it terminates; and where the value of the general multiplicator $\frac{M}{2^{n}}$ will be truly (and moſt commodiouſly) expreſſed by $\frac{1}{2}\times\frac{3}{4}\times\frac{5}{6}\ldots\ldots\times\frac{n-1}{n}$. For M being $=\frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}\ldots\ldots\overline{\frac{1}{2}n+1}}{1.\ 2\ .\ 3\ .\ 4\ \ldots\ldots\ \frac{1}{2}n}$, if the numerator hereof be multiplied by $\frac{1}{2}n.\overline{\frac{1}{2}n-1}.\overline{\frac{1}{2}n-2}.\overline{\frac{1}{2}n-3}\ldots 3.2.1$, and the denominator, at the ſame time, by its equal $1.2.3.4\ldots\ldots\overline{\frac{1}{2}n-2}.\overline{\frac{1}{2}n-1}.\frac{1}{2}n$, we ſhall then have $M=$ $\frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}\ldots\ldots 3.2.1}{1.2.3.4\ldots\frac{1}{2}n\times 1.2.3.4\ldots\frac{1}{2}n}=\frac{1.2.3.4.5.6.7\ldots\ldots\ldots\ldots n}{1.2.3.4\ldots\frac{1}{2}n\times 1.2.3.4\ldots\frac{1}{2}n}$; which divided by 2^{n} gives $\frac{M}{2^{n}}=\frac{1.2.3.4.5.6.7.8\ldots\ldots n}{2.4.6.8\ldots n\times 2.4.6.8\ldots n}=$ $\frac{1.3.5.7\ldots\ldots\overline{n-1}}{2.4.6.8\ldots\ldots n}$.

COROL-

COROLLARY II.

Hence may the fluent of $x^n\dot{x}\sqrt{1-xx}$ be likewiſe deduced; for this expreſſion may be changed to $\frac{x^n\dot{x}\times\overline{1-xx}}{\sqrt{1-xx}}$, or to $\frac{x^n\dot{x}}{\sqrt{1-xx}}-\frac{x^{n+2}\dot{x}}{\sqrt{1-xx}}$; whereof the fluent of the firſt term is already found $=\frac{1}{2}\times\frac{3}{4}\times\frac{5}{6}\ldots\ldots\frac{n-1}{n}\times$ $A-\frac{m}{m+1}\times\text{ſin.}\,2A+\frac{m}{m+1}\times\frac{m-1}{m+2}\times\frac{1}{2}\text{ſin.}\,4A$, *&c.* And, by making $n'=n+2$, and $m'=\frac{1}{2}n'\,(=m+1)$, the fluent of the ſecond term $-\frac{x^{n+2}\dot{x}}{\sqrt{1-xx}}\left(=-\frac{x^{n'}\dot{x}}{\sqrt{1-xx}}\right)$ in the ſame manner is given equal to $-\frac{1}{2}\times\frac{3}{4}\times\frac{5}{6}\ldots\ldots\frac{n'-1}{n'}\times$ $A-\frac{m'}{m'+1}\times\text{ſin.}\,2A$, *&c.* $=-\frac{1}{2}\times\frac{3}{4}\times\frac{5}{6}\ldots\ldots\frac{n+1}{n+2}\times$ $A-\frac{m+1}{m+2}\times\text{ſin.}\,2A+\frac{m+1}{m+2}\times\frac{m}{m+3}\times\frac{1}{2}\text{ſin.}\,4A$, *&c.* Whence, by adding the fluents of both terms together, we have, after proper reduction

$$\frac{1\cdot3\cdot5\cdot7\cdots\cdots\cdots\overline{n-1}}{2\cdot4\cdot6\cdot8\cdots\cdots\cdots n\cdot\overline{n+2}}$$

$$\times\begin{cases}A-\frac{m-1}{m+2}\times\text{ſ.}\,2A+\frac{m\cdot\overline{m-7}}{\overline{m+2}\cdot\overline{m+3}}\times\frac{1}{2}\text{ſ.}\,4A-\frac{m\cdot\overline{m-1}\cdot\overline{m-17}}{\overline{m+2}\cdot\overline{m+3}\cdot\overline{m+4}}\\ \times\frac{1}{3}\text{ſin.}\,6A+\frac{m\cdot\overline{m-1}\cdot\overline{m-2}\cdot\overline{m-31}}{\overline{m+2}\cdot\overline{m+3}\cdot\overline{m+4}\cdot\overline{m+5}}\times\frac{1}{4}\text{ſin.}\,8A,\ \textit{\&c.}\end{cases}$$

where the law of continuation is manifeſt, the differences (6, 10, 14, *&c.*) of the numbers 1, 7, 17, 31, *&c.* being in arithmetical progreſſion.

COROLLARY III.

Moreover, from hence the fluent of $\frac{x^n\dot{x}}{\sqrt{1-xx}}\times\overline{e+fx^2+gx^4+bx^6}$ *&c.* may be eaſily deduced: for, putting $\frac{1}{2}\times\frac{3}{4}\times\frac{5}{6}\ldots\ldots\frac{n-1}{n}$ $=$

$= q$, the fluent of the firſt term $\frac{x^n\dot{x}}{\sqrt{1-xx}}\times e$, is given, *by Corol. I*, $= qe \times \overline{\text{A} - \frac{m}{m+1}\times \text{ſin.}\,2\text{A} + \frac{m}{m+1}\times\frac{m-1}{m+2}\times\frac{1}{2}\,\text{ſin.}\,4\text{A}\ \&c.}$ and *that* of the ſecond term $\left(\frac{x^n\dot{x}}{\sqrt{1-xx}}\times fx^2\right) = \frac{n+1}{n+2}\times qf \times \overline{\text{A} - \frac{m+1}{m+2}\times \text{ſin.}\,2\text{A} + \frac{m+1}{m+2}\times\frac{m}{m+3}\times\frac{1}{2}\,\text{ſin.}\,4\text{A}\ \&c.}$ Therefore the fluent of the whole expreſſion will be found $= q$

$$\times\left\{\begin{array}{l} \overline{e\times\text{A} - \frac{m}{m+1}\times\text{ſin.}\,2\text{A} + \frac{m}{m+1}\times\frac{m-1}{m+2}\times\frac{1}{2}\,\text{ſin.}\,4\text{A}\ \&c.} \\ \frac{n+1}{n+2}f\times\overline{\text{A} - \frac{m+1}{m+2}\times\text{ſin.}\,2\text{A} + \frac{m+1}{m+2}\times\frac{m}{m+3}\times\frac{1}{2}\,\text{ſin.}\,4\text{A}\ \&c.} \\ \frac{n+1}{n+2}\times\frac{n+3}{n+4}g\times\overline{\text{A} - \frac{m+2}{m+3}\times\text{ſ.}\,2\text{A} + \frac{m+2}{m+3}\times\frac{m+1}{m+4}\times\frac{1}{2}\,\text{ſ.}\,4\text{A}\ \&c.} \\ \qquad\&c. \qquad\qquad\qquad\qquad\qquad \&c. \end{array}\right.$$

which, by making $r = \frac{n+1}{n+2}\times q$, $s = \frac{n+3}{n+4}\times r$, $t = \frac{n+5}{n+6}\times s$, *&c.* will be reduced to

$$\left\{\begin{array}{l} +\overline{eq + fr + gs + \&c.}\times\text{A} \\ -\overline{\frac{m}{m+1}.eq + \frac{m+1}{m+2}.fr + \frac{m+2}{m+3}.gs + \&c.}\times\text{ſin.}\,2\text{A} \\ +\overline{\frac{m}{m+1}.\frac{m-1}{m+2}.eq + \frac{m+1}{m+2}.\frac{m}{m+3}.fr + \frac{m+2}{m+3}.\frac{m+1}{m+4}.gs, \&c.}\times\frac{1}{2}\,\text{ſ.}\,4\text{A} \\ -\overline{\frac{m}{m+1}.\frac{m-1}{m+2}.\frac{m-2}{m+3}.eq + \frac{m+1}{m+2}.\frac{m}{m+3}.\frac{m-1}{m+4}.fr + \&c.}\times\frac{1}{3}\,\text{ſ.}\,6\text{A} \\ \qquad\&c. \qquad\qquad\qquad\qquad\qquad \&c. \end{array}\right.$$

SCHOLIUM.

From the fluents determined in the preceding Propoſition, thoſe of $\frac{z^{mp+\frac{1}{2}p-1}\dot{z}}{\sqrt{a-bz^p}}$, $\sqrt{a-bz^p}\times z^{mp+\frac{1}{2}p-1}\dot{z}$, and $\frac{z^{mp+\frac{1}{2}p-1}\dot{z}}{\sqrt{a-bz^p}}\times\overline{e+fz^p+gz^{2p}+hz^{3p}}$ *&c.* (where m denotes any whole poſitive number, and p any poſitive number whatever, whole or broken) may be eaſily deduced, by means of a proper transformation:

mation: for, $\sqrt{a-bz^p}$ being $=a^{\frac{1}{2}}\sqrt{1-\frac{bz^p}{a}}$, let there be made $\frac{bz^p}{a}=x^2$, or $z^p=\frac{a}{b}x^2$; then $z^{mp+\frac{1}{2}p}=\left.\overline{\frac{a}{b}}\right|^{m+\frac{1}{2}}\times x^{2m+1}$; and conſequently, by taking the fluxion on both ſides, $\overline{mp+\frac{1}{2}p}\,.\,z^{mp+\frac{1}{2}p-1}\dot{z}=\left.\overline{\frac{a}{b}}\right|^{m+\frac{1}{2}}\times\overline{2m+1}\,.\,x^{2m}\dot{x}$, or $z^{mp+\frac{1}{2}p-1}\dot{z}=\left.\overline{\frac{a}{b}}\right|^{m+\frac{1}{2}}\times\frac{2}{p}\,.\,x^{2m}\dot{x}$. Therefore our firſt expreſſion, $\frac{z^{mp+\frac{1}{2}p-1}\dot{z}}{\sqrt{a-bz^n}}$, will be transformed to $\frac{2a^m}{pb^{m+\frac{1}{2}}}\times\frac{x^n\dot{x}}{\sqrt{1-xx}}$ (ſuppoſing $n=2m$); whoſe fluent *(by Problem II. Corollary I.)* will be given $=$

$$\frac{1\cdot3\cdot5\cdot7\ldots\ldots\ldots\overline{n-1}}{2\cdot4\cdot6\cdot8\ldots\ldots\ldots n}\times\frac{2a^m}{pb^{m+\frac{1}{2}}}\times\overline{A-\frac{m}{m+1}\text{ſ}.2A+\frac{m}{m+1}\cdot\frac{m-1}{m+2}\times\tfrac{1}{2}\text{ſ}.4A-\frac{m}{m+1}\cdot\frac{m-1}{m+2}\cdot\frac{m-2}{m+3}\times\tfrac{1}{3}\text{ſ}.6A}$$

$+$ &c. where A repreſents the arch whoſe ſine is $\sqrt{\frac{bz^p}{a}}$, the radius being unity.

In the ſame manner the fluent of our ſecond expreſſion, $\sqrt{a-bz^p}\times z^{mp+\frac{1}{2}p-1}\dot{z}$, will be given *(by Corol. II.)* equal to

$$\frac{1\cdot3\cdot5\cdot7\ldots\ldots\ldots\overline{n-1}}{2\cdot4\cdot6\cdot8\ldots\ldots\ldots\overline{n+2}}\times\frac{2a^{m+1}}{pb^{m+\frac{1}{2}}}\times\left\{\begin{array}{l}A-\frac{m-1}{m+2}\text{ſin.}\,2A+\frac{m\,.\,\overline{m-7}}{\overline{m+2}\,.\,\overline{m+3}}\times\frac{1}{2}\text{ſin.}\,4A-\frac{m\,.\,\overline{m-1}\,.\,\overline{m-17}}{\overline{m+2}\,.\,\overline{m+3}\,.\,\overline{m+4}}\\ \times\frac{1}{3}\text{ſin.}\,6A+\frac{m\,.\,\overline{m-1}\,.\,\overline{m-2}\,.\,\overline{m-31}}{\overline{m+2}\,.\,\overline{m+3}\,.\,\overline{m+4}\,.\,\overline{m+5}}\times\frac{1}{4}\text{ſin.}\,8A-\text{\&c.}\end{array}\right.$$

Laſtly, $\frac{z^{mp+\frac{1}{2}p-1}\dot{z}}{\sqrt{a-bz^p}}\times\overline{e+fz^p+gz^{2p}+bz^{3p}\text{ \&c.}}$ will be transformed to $\frac{2a^m}{pb^{m+\frac{1}{2}}}\times\frac{x^n\dot{x}}{\sqrt{1-xx}}\times\overline{e+\frac{fa}{b}x^2+\frac{ga^2}{bb}x^4+\text{\&c.}}$ and the fluent thereof *(by Corol. III.)* will therefore be given equal to

$$\frac{2a^m}{pb^{m+\frac{1}{2}}}\times\left\{\begin{array}{l}\overline{+eq+fr\times\frac{a}{b}+gs\times\frac{a^2}{b^2}+\text{\&c.}}\times A\\ \overline{-eq\times\frac{m}{m+1}+fr\times\frac{a}{b}\times\frac{m+1}{m+2}+gs\times\frac{a^2}{b^2}\times\frac{m+2}{m+3}\text{\&c.}}\times\text{ſ}.2A\end{array}\right.$$

$$\frac{2a^m}{pb^{m+\frac{1}{2}}}\times\overline{eq\times\underset{\&c.}{\frac{m}{m+1}\times\frac{m-1}{m+2}}+fr\times\frac{a}{b}\times\underset{\&c.}{\frac{m+1}{m+2}\times\frac{m}{m+3}}\ \&c.}\times\tfrac{1}{2}\,\text{ſ}.4A$$

wherein $q=\frac{1\cdot3\cdot5\cdot7\ldots\ldots\overline{n-1}}{2\cdot4\cdot6\cdot8\ldots\ldots\ n}$, $r=q\times\frac{n+1}{n+2}$, $s=r\times\frac{n+3}{n+4}$, $t=s\times\frac{n+5}{n+6}$, &c.

When $\sqrt{\frac{bz^n}{a}}$ becomes equal to the radius, or unity, A will be an arch of 90 degrees; and therefore, the ſines of all the arches 2A, 4A, 6A, &c. being then equal to nothing, the fluent of $\frac{z^{mp+\frac{1}{2}p-1}\dot{z}}{\sqrt{a-bz^n}}$ will, in that circumſtance, be barely $=$ $\frac{1\cdot3\cdot5\cdot7\ldots\ldots\overline{n-1}}{2\cdot4\cdot6\cdot8\ldots\ldots\overline{n-2}}\times\frac{2a^m}{pb^{m+\frac{1}{2}}}\times A$. Moreover, the fluent of $\sqrt{a-bz^p}\times z^{mp+\frac{1}{2}p-1}\dot{z}$ will then become $\frac{1\cdot3\cdot5\cdot7\ldots\ldots\overline{n-1}}{2\cdot4\cdot6\cdot8\ldots\ldots\overline{n+2}}$ $\times\frac{2a^{m+1}}{pb^{m+\frac{1}{2}}}\times A$; and that of $\frac{\dot{z}^{mp+\frac{1}{2}p-1}\dot{z}}{\sqrt{a-bz^p}}\times\overline{e+fz^p+gz^{2p}+hz^{3p}\ \&c.}=e+\frac{n+1}{n+2}\cdot\frac{fa}{b}+\frac{n+1}{n+2}\cdot\frac{n+3}{n+4}\cdot\frac{ga^2}{b^2}+\frac{n+1}{n+2}\cdot\frac{n+3}{n+4}\cdot\frac{n+5}{n+6}\cdot\frac{ha^3}{b^3}\ \&c.$ $\times\frac{2qa^m}{pb^{m+\frac{1}{2}}}\times A$; where $q=\frac{1\cdot3\cdot5\ldots\ldots\overline{n-1}}{2\cdot4\cdot6\ldots\ldots\ n}$, and where, if $m=0$, q muſt be taken $=1$.

PROBLEM III.

To determine the fluent of $\dot{z}\times$ *coſ.* $mz\times$ *coſ.* $nz\times$ *coſ.* pz &c. *in which* mz, nz, pz, &c. *are any given multiples of the arch* z; *the radius of the circle being unity.*

Make $A=mz$, $B=nz$, $C=pz$, &c. and find *(by the method on p.* 81) a ſeries of co-ſines of the multiples of z, to expreſs the continual product (coſ. A × coſ. B × coſ. C, &c.) of the co-ſines propounded; which ſeries let be denoted by $a\times\overline{\text{coſ}.\,\alpha z+\text{coſ}.\,\beta z+\text{coſ}.\,\gamma z+\text{coſ}.\,\delta z\ \&c.}$ (a, α, β, &c. being conſtant quantities): then will our given expreſſion become

$a\dot{z} \times \overline{\text{coſ.}\,\alpha z + \text{coſ.}\,\beta z + \text{coſ.}\,\gamma z + \text{coſ.}\,\delta z\ \&c.}$ and its fluent will therefore (by proceeding as in Prob. II.) appear to be $= a$ into $\frac{\text{ſin.}\,\alpha z}{\alpha} + \frac{\text{ſin.}\,\beta z}{\beta} + \frac{\text{ſin.}\,\gamma z}{\gamma} + \frac{\text{ſin.}\,\delta z}{\delta}$ &c.

Thus, for example, let the fluent of $\dot{z} \times \text{coſ.}\,mz \times \text{coſ.}\,nz$ be required; then will $\text{coſ.}\,A \times \text{coſ.}\,B$ $(= \text{coſ.}\,mz \times \text{coſ.}\,nz) = \frac{1}{2} \times \overline{\text{coſ.}\,\overline{A+B} + \text{coſ.}\,\overline{A-B}} = \frac{1}{2} \times \overline{\text{coſ.}\,\overline{m+n}.z + \overline{m-n}.z}$: therefore, in this caſe, $a = \frac{1}{2}$, $\alpha = m + n$, and $\beta = m - n$; and conſequently the fluent ſought $= \frac{1}{2}$ into $\frac{\text{ſin.}\,\overline{m+n}.z}{m+n} + \frac{\text{ſin.}\,\overline{m-n}.z}{m-n}$.

In like manner, if the fluent of $\dot{z} \times \textit{coſ.}\,mz \times \textit{coſ.}\,nz \times \textit{coſ.}\,pz$ were to be required; then would $\text{coſ.}\,mz \times \text{coſ.}\,nz \times \text{coſ.}\,pz$ $(= \text{coſ.}\,A \times \text{coſ.}\,B \times \text{coſ.}\,C) = \frac{1}{4}$ into $\text{coſ.}\,\overline{m+n+p}.z + \text{coſ.}\,\overline{m+n-p}.z + \text{coſ.}\,\overline{m-n+p}.z + \text{coſ.}\,\overline{-m+n+p}.z$; and therefore the fluent ſought $= \frac{1}{4}$ into $\frac{\text{ſin.}\,\overline{m+n+p}.z}{m+n+p} + \frac{\text{ſin.}\,\overline{m+n-p}.z}{m+n-p} + \frac{\text{ſin.}\,\overline{m-n+p}.z}{m-n+p} + \frac{\text{ſin.}\,\overline{-n+m+p}.z}{-n+m+p}$.

By the very ſame method the fluent may be determined when ſome, or all of the factors are ſines (inſtead of co-ſines). Thus, if there be given $\dot{z} \times \text{coſ.}\,mz \times \text{ſin.}\,nz$, it may be wrote $\dot{z} \times \text{coſ.}\,mz \times \text{coſ.}\,\overline{90^\circ - nz}$; which is $= \frac{\dot{z}}{2}$ into $\text{coſ.}\,90^\circ + \overline{m-n}.z + \text{coſ.}\,\overline{m+n}.z - 90^\circ = \frac{\dot{z}}{2}$ into $\text{ſin.}\,\overline{n-m}.z + \text{ſin.}\,\overline{n+m}.z$; and ſo the fluent (by proceeding as in *Problem I.*) will come out $= \frac{1}{2}$ into $\frac{\text{verſed-ſin.}\,\overline{n-m}.z}{n-m} + \frac{\text{verſed-ſin.}\,\overline{n+m}.z}{n+m}$.

PROBLEM IV.

From the equation $ay + \frac{b\dot{y}}{\dot{z}} + \frac{c\ddot{y}}{\dot{z}^2} + \frac{d\dddot{y}}{\dot{z}^3} + \frac{e\ddddot{y}}{\dot{z}^4} +$ *&c.* $= 0$ *(wherein* a, b, c, d, &c. *denote conſtant quantities) ; it is propoſed to find the value of* y *in terms of* z.

Aſſume $y = \alpha M^{mz} + \beta M^{nz} + \gamma M^{pz} + \delta M^{qz}$ *&c.* in which M denotes the number whoſe hyperbolical logarithm is *unity* : then will

$$\begin{aligned} \dot{y} &= m\dot{z}\alpha M^{mz} + n\dot{z}\beta M^{nz} + p\dot{z}\gamma M^{pz} \text{ \&c.} \\ \ddot{y} &= m^2\dot{z}^2\alpha M^{mz} + n^2\dot{z}^2\beta M^{nz} + p^2\dot{z}^2\gamma M^{pz} \text{ \&c.} \\ \dddot{y} &= m^3\dot{z}^3\alpha M^{mz} + n^3\dot{z}^3\beta M^{nz} + p^3\dot{z}^3\gamma M^{pz} \text{ \&c.} \\ &\text{\&c.} \qquad\qquad\qquad\qquad \text{\&c.} \end{aligned}$$

Which values being ſubſtituted in the given equation, we have

$$\left.\begin{array}{r} a\alpha M^{mz} + a\beta M^{nz} + a\gamma M^{pz} \text{ \&c.} \\ bm\alpha M^{mz} + bn\beta M^{nz} + bp\gamma M^{pz} \text{ \&c.} \\ cm^2\alpha M^{mz} + cn^2\beta M^{nz} + cp^2\gamma M^{pz} \text{ \&c.} \\ dm^3\alpha M^{mz} + dn^3\beta M^{nz} + dp^3\gamma M^{pz} \text{ \&c.} \end{array}\right\} = 0$$

$$\text{\&c.} \qquad\qquad \text{\&c.}$$

From whence, by equating the homologous terms, we have $a + bm + cm^2 + dm^3$ *&c.* $= 0$, $a + bn + cn^2 + dn^3$ *&c.* $= 0$, $a + bp + cp^2 + dp^3$ *&c.* $= 0$, *&c.* that is, the required values of m, n, p, &c. will always be the roots of an equation, $a + bx + cx^2 + dx^3$ *&c.* $= 0$, wherein the given quantities are the ſame, in every term, with thoſe in the fluxional equation propounded. Therefore, when theſe roots are known, the value of y will alſo be known : in which the coefficients α, β, γ, δ, *&c.* may denote any conſtant quantities at pleaſure ; as is evident from the proceſs.

When ſome of the roots of the equation $a + bx + cx^2 + dx^3 + ex^4$ *&c.* $= 0$, happen to be impoſſible, the values of the correſponding terms of the ſeries $\alpha M^{mz} + \beta M^{nz} + \gamma M^{pz} + \delta M^{qz}$ *&c.* will then be expreſſed by means of the ſines and co-ſines of circular arcs. Thus, for example, let the fluxionary equation propounded be $y - \frac{\ddddot{y}}{\dot{z}^4} = 0$; then we ſhall have $1 - x^4 = 0$; whereof the four roots are 1, -1, $+\sqrt{-1}$, and $-\sqrt{-1}$; and, theſe being ſubſtituted for m, n, p, and

q, reſpectively, y will here become $= \alpha M^{z} + \beta M^{-z} + \gamma M^{+z\sqrt{-1}} + \delta M^{-z\sqrt{-1}}$. Now, to take away the imaginary terms $\gamma M^{z\sqrt{-1}} + \delta M^{-z\sqrt{-1}}$, we may write $k + l = \gamma$, and $k - l = \delta$; whereby the ſum of the ſaid terms will be $= k \times \overline{M^{z\sqrt{-1}} + M^{-z\sqrt{-1}}} + l \times \overline{M^{z\sqrt{-1}} - M^{-z\sqrt{-1}}} = 2k \times \text{coſ.}\, z + \frac{2l}{\sqrt{-1}} \times \text{ſin.}\, z$ *(vid. p.* 78) : whence (putting $h = \frac{l}{\sqrt{-1}}$) we have $y = \alpha M^{z} + \beta M^{-z} + 2k \times \text{coſ.}\, z + 2h \times \text{ſin.}\, z$; where α, β, h, and k denote any conſtant quantities, at pleaſure.

In like manner, ſuppoſing the equation given to be $y + \frac{d\dddot{y}}{\dot{z}^3} = 0$, we ſhall have $1 + dx^3 = 0$; whereof the three roots are $-d^{-\frac{1}{3}}$, $d^{-\frac{1}{3}} \times \overline{\frac{1}{2} + \sqrt{-\frac{3}{4}}}$, and $d^{-\frac{1}{3}} \times \overline{\frac{1}{2} - \sqrt{-\frac{3}{4}}}$; which, if s be put $= d^{-\frac{1}{3}}$, and $t = d^{-\frac{1}{3}} \times \sqrt{\frac{3}{4}}$, will be more commodiouſly expreſſed by $-s$, $\frac{1}{2}s + t\sqrt{-1}$, and $\frac{1}{2}s - t\sqrt{-1}$: And theſe values being ſubſtituted in the room of m, n, and p, we ſhall have $y = \alpha M^{-sz} + \beta M^{\frac{1}{2}sz + tz\sqrt{-1}} + \gamma M^{\frac{1}{2}sz - tz\sqrt{-1}} = \alpha M^{-sz} + M^{\frac{1}{2}sz} \times \overline{\beta M^{tz\sqrt{-1}} + \gamma M^{-tz\sqrt{-1}}}$; which, by reaſoning as in the preceding caſe, is reduced to $y = \alpha M^{-sz} + M^{\frac{1}{2}sz} \times \overline{2h \times \text{ſin.}\, tz + 2k \times \text{coſ.}\, tz}$.

PROBLEM V.

From the equation $ay + \frac{b\dot{y}}{\dot{z}} + \frac{c\ddot{y}}{\dot{z}^2} + \frac{d\dddot{y}}{\dot{z}^3}$ *&c.* $= AM^{pz} + BM^{qz} + CM^{rz}$ &c. *to determine the value of* y; *ſuppoſing* M *to denote the number whoſe hyperbolical logarithm is unity, and* a, b, c, &c. A, B, C, *&c. any conſtant quantities.*

Aſſuming $y = PM^{pz} + QM^{qz} + RM^{rz}$ *&c.* we have

$$\dot{y} = p\dot{z}PM^{pz} + q\dot{z}QM^{qz} + r\dot{z}RM^{rz} \text{ \&c.}$$
$$\ddot{y} = p^2\dot{z}^2BM^{pz} + q^2\dot{z}^2QM^{qz} + r^2\dot{z}^2RM^{rz} \text{ \&c.}$$
$$\text{\&c.} \qquad\qquad \text{\&c.}$$

which values being ſubſtituted in the given equation, it becomes

$$\left.\begin{array}{llll} aPM^{pz} + & aQM^{qz} + & aRM^{rz} & \&c. \\ bpPM^{pz} + & bqQM^{qz} + & brRM^{rz} & \&c. \\ cp^2PM^{pz} + & cq^2QM^{qz} + & cr^2RM^{rz} & \&c. \\ \quad\&c. & & \&c. & \end{array}\right\} = \begin{array}{l} AM^{pz} + BM^{qz} + \\ \quad CM^{rz}\ \&c. \end{array}$$

From whence, by comparing the homologous terms, we have $P = \frac{A}{a + bp + cp^2 + dp^3\ \&c.}$, $Q = \frac{B}{a + bq + cq^2 + dq^3\ \&c.}$, $R = \frac{C}{a + br + cr^2 + dr^3\ \&c.}$, &c. whereby one value of y is known. But the value or fluent thus found, in order to render it general, muſt be corrected by the value of y found in the preceding Problem, that is, by the quantity $\alpha M^{m'z} + \beta M^{n'z} + \gamma M^{p'z}$ &c. wherein m', n', p', &c. denote the roots of the equation $a + bx + cx^2 + dx^3$ &c. $= 0$, and α, β, γ, &c. any conſtant quantities. For, ſince all the terms ariſing from this laſt part of the value of y, by ſubſtituting in the given equation, do mutually deſtroy one another, the other terms affected with P, Q, R, &c, will be no-ways influenced thereby, but remain exactly the ſame as above determined.

COROLLARY.

If the equation given be $m^2y + \frac{\ddot{y}}{\dot{z}\dot{z}} = AM^{pz} + BM^{qz} + CM^{rz} + DM^{sz}$ &c. then (a being $= m^2$, $c = 1$, and b, d, e, &c. each $= 0$) we have $P = \frac{A}{mm + pp}$, $Q = \frac{B}{mm + qq}$, &c. and $x = \pm m\sqrt{-1}$; and conſequently $y = \alpha M^{mz\sqrt{-1}} + \beta M^{-mz\sqrt{-1}} + \frac{AM^{pz}}{mm + pp} + \frac{BM^{qz}}{mm + qq}$ &c. $= 2h \times \text{ſin.}\, mz + 2k \times \text{coſ.}\, mz + \frac{AM^{pz}}{mm + pp} + \frac{BM^{qz}}{mm + qq}$ &c. (*ſee Ex. I. to Prob. V.*)

Hence it follows, that, if the equation given be $m^2y + \frac{\ddot{y}}{\dot{z}\dot{z}}$ $(= AM^{\pi z\sqrt{-1}} + AM^{-\pi z\sqrt{-1}} + A'M^{\varrho z\sqrt{-1}} + A'M^{-\varrho z\sqrt{-1}}$ &c.$)$ $= 2A \times \text{coſ.}\, \pi z + 2A' \times \text{coſ.}\, \varrho z$ &c. the value of y (by ſubſtituting $-\pi^2 = p^2$, $-\pi^2 = q^2$, $-\varrho^2 = r^2$, $-\varrho^2 = s^2$, $A = B$, $A' = C$, $A' = D$, &c.) will come out $= 2h \times \text{ſin.}\, mz$

+

$+ 2k \times \text{coſ.}\, mz + \frac{AM^{\pi z\sqrt{-1}}}{mm - \pi\pi} + \frac{AM^{-\pi z\sqrt{-1}}}{mm - \pi\pi}$ &c. $= 2h \times \text{ſin.}\, mz$
$+ 2k \times \text{coſ.}\, mz + \frac{2A}{mm - \pi\pi} \times \text{coſ.}\, \pi z + \frac{2A'}{mm - \varrho\varrho} \times \text{coſ.}\, \varrho z$ &c. Which equation (wherein h and k may denote any conſtant quantities) is of ſingular uſe in determining the figure of the lunar orbit.

In like manner, when the general equation propounded is of this form, $ay + \frac{b\dot{y}}{\dot{z}} + \frac{c\ddot{y}}{\dot{z}^2} + \frac{d\dddot{y}}{\dot{z}^3}$ &c. $= Az^v + Bz^{v-1} + Cz^{v-2} + Dz^{v-3}$ &c. the value of y may be determined, by aſſuming $Pz^v + Qz^{v-1} + Rz^{v-2}$ &c. $= y$; from whence, by ſubſtituting in the given equation, and comparing the homologous terms, there will be had $P = \frac{A}{a}$, $Q = \frac{B - vbP}{a}$, $R = \frac{C - \overline{v-1}\,.\,bQ - v\,.\,\overline{v-1}\,.\,cP}{a}$, $S = \frac{D - \overline{v-2}\,.\,bR - \overline{v-1}\,.\,\overline{v-2}\,.\,cQ - v\,.\,\overline{v-1}\,.\,\overline{v-2}\,.\,dP}{a}$, &c. where the ſeries will always terminate, provided v is any poſitive integer; and where, if to the value of y thus determined, the correction or ſeries ($\alpha M^{mz} + \beta M^{nz} + \gamma M^{pz}$ &c.) found by Prob. IV. be added, the general value of y will be obtained.

PROBLEM VI.

To determine the value of y in any fluxionary equation of this form, $\frac{\ddddot{y}}{\dot{z}^4} + \frac{a\dddot{y}}{\dot{z}^3} + \frac{b\ddot{y}}{\dot{z}^2} + \frac{c\dot{y}}{\dot{z}} + dy = \Delta$; *ſuppoſing* Δ *to repreſent any quantity expreſſed in terms of z and known coefficients.*

1°. Make $y = M^{pz} \times \text{flu.}\, \dot{z}PM^{-pz}$ (wherein P denotes a variable quantity, and p a conſtant one, to be determined): ſo ſhall $y \times M^{-pz} = \text{flu.}\, \dot{z}PM^{-pz}$, or (by taking the fluxions) $\dot{y} \times M^{-pz} - y \times p\dot{z}M^{-pz} = \dot{z}PM^{-pz}$; whence, dividing the whole by $\dot{z}M^{-pz}$, we have $\frac{\dot{y}}{\dot{z}} - py = P$.

2°. Make $P = M^{qz} \times \underline{\text{flu.}\, \dot{z}QM^{-qz}}$; then our laſt equation will be transformed to $\frac{\dot{y}}{\dot{z}} - py \times M^{-qz} = \text{flu.}\, \dot{z}QM^{-qz}$;

whence, by taking the fluxions, $\overline{\frac{\ddot{y}}{\dot{z}} - p\dot{y}} \times M^{-qz} + \overline{\frac{\dot{y}}{\dot{z}} - py} \times -q\dot{z}M^{-qz} = \dot{z}QM^{-qz}$, or $\frac{\ddot{y}}{\dot{z}^2} - \overline{p+q}\cdot\frac{\dot{y}}{\dot{z}} + pqy = Q$, by dividing the whole by $\dot{z}M^{qz}$.

3°. Make, now, $Q = M^{rz} \times$ flu. $\dot{z}RM^{-rz}$; then will $\overline{\frac{\ddot{y}}{\dot{z}^2} - \overline{p+q}\cdot\frac{\dot{y}}{\dot{z}} + pqy} \times M^{-rz} =$ fluent of $\dot{z}RM^{-rz}$, or $\overline{\frac{\dddot{y}}{\dot{z}^3} - \overline{p+q}\cdot\frac{\ddot{y}}{\dot{z}} + pq\dot{y}} \times M^{-rz} + \overline{\frac{\ddot{y}}{\dot{z}^2} - \overline{p+q}\cdot\frac{\dot{y}}{\dot{z}} + pqy} \times -r\dot{z}M^{-rz} = \dot{z}RM^{-rz}$, or, laſtly, $\frac{\dddot{y}}{\dot{z}^3} - \overline{p+q+r}\cdot\frac{\ddot{y}}{\dot{z}^2} + \overline{pq+pr+qr}\cdot\frac{\dot{y}}{\dot{z}} - pqry = R$.

4°. Make, again, $R = M^{sz} \times$ flu. $\dot{z}SM^{-sz}$, and proceed in the ſame manner; ſo ſhall $\frac{\ddddot{y}}{\dot{z}^4} - \overline{p+q+r+s}\cdot\frac{\dddot{y}}{\dot{z}^3} - \overline{pq+pr+ps+qr+qs+rs}\cdot\frac{\ddot{y}}{\dot{z}^2} - \overline{pqr+pqs+prs+qrs}\cdot\frac{\dot{y}}{\dot{z}} + pqrsy = S$: from whence the law of continuation is manifeſt.

Let, now, the ſeveral terms of the equation $\frac{\ddddot{y}}{\dot{z}^4} - \overline{p+q+r+s}\cdot\frac{\dddot{y}}{\dot{z}^3}$ &c. $= S$ be compared with the correſponding terms of the given one, $\frac{\ddddot{y}}{\dot{z}^4} + \frac{a\dddot{y}}{\dot{z}^3}$ &c. $= \Delta$: ſo ſhall $p + q + r$ &c. $= -a$, $pq + pr + ps + qr$ &c. $= -b$, $pqr + pqs$ &c. $= -c$, &c. &c. Whence, from the *geneſis* of equations, it is evident, that p, q, r, &c. are the roots of an equation $x^4 + ax^3 + bx^2 + cx + d = 0$ (or, $x^n + ax^{n-1} + bx^{n-2}$ &c. $= 0$) wherein the given quantities are the very ſame with thoſe in the equation propounded. Therefore, when the values of theſe roots are found (by any of the known methods) the values of R, Q, P, and y may alſo be found, one from another, ſucceſſively. *Q. E. I.*

The ſame otherwiſe.

Let (if poſſible) $y = AM^{pz} \times \text{flu.}\, \dot{z}\Delta M^{-pz} + BM^{qz} \times \text{flu.}\, \dot{z}\Delta M^{-qz} + CM^{rz} \times \text{flu.}\, \dot{z}\Delta M^{-rz}$ &c. (A, B, C, *&c.* p, q, r, *&c.* being conſtant quantities to be determined): then, by taking the fluxions, we ſhall have

$$\frac{\dot{y}}{\dot{z}} = pAM^{pz} \times \text{flu.}\, \dot{z}\Delta M^{-pz} + A\Delta + qBM^{qz} \times \text{flu.}\, \dot{z}\Delta M^{-qz} + \&c.$$

$$\frac{\ddot{y}}{\dot{z}^2} = p^2 AM^{pz} \times \text{fl.}\, \dot{z}\Delta M^{-pz} + pA\Delta + \frac{A\dot{\Delta}}{\dot{z}} + q^2 BM^{qz} \times \text{fl.}\, \dot{z}\Delta M^{-qz}\ \&c.$$

$$\frac{\dddot{y}}{\dot{z}^3} = p^3 AM^{pz} \times \text{flu.}\, \dot{z}\Delta M^{-pz} + p^2 A\Delta + \frac{pA\dot{\Delta}}{\dot{z}} + \frac{A\ddot{\Delta}}{\dot{z}\dot{z}} + \&c.$$

$$\frac{\ddddot{y}}{\dot{z}^4} = p^4 AM^{pz} \times \text{flu.}\, \dot{z}\Delta M^{-pz} + p^3 A\Delta + \frac{p^2 A\dot{\Delta}}{\dot{z}} + \frac{pA\ddot{\Delta}}{\dot{z}\dot{z}} + \frac{A\dddot{\Delta}}{\dot{z}\dot{z}\dot{z}}\ \&c.$$

Which values being ſubſtituted in the given equation, and the homologous terms being compared, we ſhall thereby get $p^4 + ap^3 + bp^2 + cp + d = 0$, $q^4 + aq^3 + bq^2 + cq + d = 0$, *&c.*

$$\text{alſo} \left\{ \begin{array}{l} \overline{p^3 + ap^2 + bp + c} \times A + \overline{q^3 + aq^2 + bq + c} \times B \\ \overline{r^3 + ar^2 + br + c} \times C + \overline{s^3 + as^2 + bs + c} \times D \end{array} \right\} = 1,$$

$$\overline{p^2 + ap + b} \times A + \overline{q^2 + aq + b} \times B + \overline{r^2 + ar + b} \times C + \overline{s^2 + as + b} \times D = 0$$

$$\overline{p + a} \times A + \overline{q + a} \times B + \overline{r + a} \times C + \overline{s + a} \times D = 0$$

$$A + B + C + D = 0.$$

Now, from the former of theſe equations, $p^4 + ap^3 + bp^2 + cp + d = 0$, $q^4 + aq^3 + bq^2 + cq + d = 0$, *&c.* it appears evident, that p, q, r, &c. are the roots of the equation $x^4 + ax^3 + bx^2 + cx + d = 0$ (or, more generally, of $x^n + ax^{n-1} + bx^{n-2}$ *&c.* $= 0$, n denoting the order to which the fluxions aſcend in the given equation); which roots being therefore found (by any of the known methods) the values of p, q, r, &c. will be obtained. But to find from thence and the remaining equations, the values of A, B, C, *&c.* let the laſt of theſe equations multiplied by a, be ſubtracted from the preceding one, ſo ſhall $pa + qB + rC + sD = 0$: moreover, let this new equation multiplied by a, be ſubtracted from the laſt but two, and from the remainder let $bA + bB + bC + bD = 0$ be again ſubtracted, whence will be had $p^2A + q^2B$ +

$+ r^2C + s^2D = 0$: and, in the ſame manner, from the firſt equation, will be had $p^3A + q^3B + r^3C + s^3D = 1$ (becauſe 1, and not 0, forms the latter part of that equation).

Now, from each of the equations ($A + B + C + D = 0$, $pA + qB + rC + sD = 0$, $p^2A + q^2B + r^2C + s^2D = 0$, $p^3A + q^3B + r^3C + s^3D = 1$) thus derived, let the preceding one multiplied by p, be ſubtracted: ſo ſhall

$$\overline{q-p}\,.\,B + \overline{r-p}\,.\,C + \overline{s-p}\,.\,D = 0,$$
$$\overline{q-p}\,.\,qB + \overline{r-p}\,.\,rC + \overline{s-p}\,.\,sD = 0,$$
$$\overline{q-p}\,.\,q^2B + \overline{r-p}\,.\,r^2C + \overline{s-p}\,.\,s^2D = 1.$$

Moreover, from each of theſe laſt equations, let the preceding one multiplied by q, be in like manner ſubtracted; whence will be had

$$\overline{r-p}\,.\,\overline{r-q}\,.\,C + \overline{s-p}\,.\,\overline{s-q}\,.\,D = 0,$$
$$\overline{r-p}\,.\,\overline{r-q}\,.\,rC + \overline{s-p}\,.\,\overline{s-q}\,.\,sD = 1.$$

Again, from the laſt of theſe, let the preceding one multiplied by r, be ſubtracted; then will $\overline{s-p}\,.\,\overline{s-q}\,.\,\overline{s-r}\,.\,D = 1$, and conſequently $D = \frac{1}{\overline{s-p}\,.\,\overline{s-q}\,.\,\overline{s-r}}$: whence it is manifeſt by inſpection (becauſe p is the ſame with reſpect to A, as s is to D, &c.) that $A = \frac{1}{\overline{p-q}\,.\,\overline{p-r}\,.\,\&c.}$, $B = \frac{1}{\overline{q-p}\,.\,\overline{q-r}\,.\,\&c.}$, $C = \frac{1}{\overline{r-p}\,.\,\overline{r-q}\,.\,\&c.}$, &c. From whence the value of y ($= AM^{pz} \times \text{flu.}\, \dot{z}\Delta M^{-pz} + BM^{qz} \times \text{flu.}\, \dot{z}\Delta M^{-qz}$ &c.) will be known, let the orders of fluxions in the equation aſcend to what height they will.—Thus, for example, let the equation propounded be $\frac{\ddot{y}}{\dot{z}^2} + m^2y = M^{\pi z}$: in which caſe, a being $= 0$, $b = m^2$, $c = 0$, &c. our general equation, $x^n + ax^{n-1} + bx^{n-2} + cx^{n-3}$ &c. $= 0$, will therefore become $x^2 + m^2 = 0$; whereof the two roots (p and q) are $m\sqrt{-1}$, and $-m\sqrt{-1}$; from whence A $\left(= \frac{1}{p-q}\right) = \frac{1}{2m\sqrt{-1}}$, and B $\left(= \frac{1}{q-p}\right) = -\frac{1}{2m\sqrt{-1}}$: alſo, becauſe Δ is here $= M^{\pi z}$, we have $y = AM^{pz}$

×

$\times$ flu. $\dot{z}M^{\pi z - pz} + BM^{qz} \times$ flu. $\dot{z}M^{\pi z - qz} = \frac{AM^{\pi z}}{\pi - p} + \frac{BM^{\pi z}}{\pi - q} = \frac{M^{\pi z}}{\pi^2 + p^2}$ (by ſubſtituting the values of p, q, A, and B). But in order to render the ſolution general, the value of y thus found muſt, always, be corrected, or augmented by the quantity $\alpha M^{pz} + \beta M^{qz} + \gamma M^{rz}$ *&c.* (given by Prob. IV.) where α β, γ, δ, *&c.* may denote any conſtant quantities whatever, poſitive, or negative.—Other inſtances of the uſe of exponential quantities, and of the meaſures of angles and ratios, in the reſolution of fluxionary equations of different kinds, might be given; but I ſhall conclude here, with obſerving, that Δ, in this laſt ſolution, may denote any quantity wherein both y and z enter, as well as one in which z is alone concerned independent of y.

An INVESTIGATION of a GENERAL RULE for the Reſolution of Iſoperimetrical Problems of all Orders.

LEMMA.

SUPPOSING α, β, γ, δ, ε, *&c. to be a ſeries of indeterminate quantities,*

$$\textit{and that}\left\{\begin{array}{llll} Q, & R, & S, & T \\ Q', & R', & S', & T' \\ Q'', & R'', & S'', & T'' \\ Q''', & R''', & S''', & T''' \\ Q'''', & R'''', & S'''', & T'''' \\ \&c. & & & \end{array}\right\}\begin{array}{l}\textit{are any quanti-}\\ \textit{ties compoſed of}\end{array}\left\{\begin{array}{c} \alpha \\ \beta \\ \gamma \\ \delta \\ \varepsilon \\ \&c. \end{array}\right\}\begin{array}{l}\textit{and given}\\ \textit{quantities};\end{array}$$

It is propoſed to find an equation for the relation of α, β, γ, δ, *&c. ſo that the quantity* $Q + Q' + Q'' + Q''' + Q''''$ *&c. ſhall be a maximum* or *minimum, at the ſame time that the other quantities* $R + R' + R'' + R''' + R''''$ *&c.* $S + S' + S' + S'' + S''' + S''''$ *&c. and* $T + T' + T'' + T''' + T''''$ *&c. are all of them given, or ſuppoſed to remain invariable.*

Let $\bar{Q}$, $\bar{R}$, $\bar{S}$, $\bar{T}$ denote any correſponding terms of the ſeries's $Q + Q' + Q'' + Q''' + Q''''$ *&c.* $R + R' + R'' + R''' + R''''$ *&c.* $S + S' + S'' + S''' + S''''$ *&c.* $T + T' + T'' + T''' + T''''$ *&c.* reſpectively, expreſſed in terms of u, any one of the propoſed quantities α, β, γ, δ, ε *&c.* moreover let the fluxion of Q (α being variable) be denoted by $q\dot{\alpha}$; that of R by $r\dot{\alpha}$; that of Q' by $q'\dot{\beta}$; that R' by $\gamma'\dot{\beta}$, *&c. &c.*

It is evident that the quantity $Q + Q' + Q'' + Q''' + Q''''$ *&c.* cannot be a *maximum* or *minimum*, when $R + R' + R'' + R''' + R''''$ *&c.* $S + S' + S'' + S''' + S''''$ &c. and $T + T' + T'' + T''' + T''''$ &c. are given quantities, unleſs the part $\bar{Q} + Q + Q' + Q''$ is a *maximum* or *minimum*, when the parts $\bar{R} + R + R' + R''$, $\bar{S} + S + S' + S''$, and $\bar{\mathcal{T}} + \mathcal{T} + \mathcal{T}' + \mathcal{T}''$ are given quantities; becauſe the terms in theſe parts may be alone made variable, while the other terms are ſuppoſed to remain the ſame, whereby the whole ſums, $R + R' + R'' + R''' + R''''$ &c. $S + S' + S'' + S''' + S''''$ &c. will remain the ſame, and the quantity $\mathcal{Q} + \mathcal{Q}' + \mathcal{Q}'' + \mathcal{Q}''' + \mathcal{Q}''''$ &c. will be a *maximum* or *minimum*, when the part $\bar{\mathcal{Q}} + \mathcal{Q} + \mathcal{Q}' + \mathcal{Q}''$ is ſo. But when $\bar{\mathcal{Q}} + \mathcal{Q} + \mathcal{Q}' + \mathcal{Q}''$ is a *maximum* or *minimum*, and $\bar{R} + R + R' + R''$, $\bar{S} + S + S' + S''$, and $\bar{\mathcal{T}} + \mathcal{T} + \mathcal{T}' + \mathcal{T}''$ are given (or conſtant) quantities, their fluxions will be, all of them, equal to nothing; whence we have theſe equations,

$\bar{q}\dot{u} + q\dot{\alpha} + q'\dot{\beta} + q''\dot{\gamma} = 0$

$\bar{r}\dot{u} + r\dot{\alpha} + r'\dot{\beta} + r''\dot{\gamma} = 0$

$\bar{s}\dot{u} + s\dot{\alpha} + s'\dot{\beta} + s''\dot{\gamma} = 0$

$\bar{t}\dot{u} + t\dot{\alpha} + t'\dot{\beta} + t''\dot{\gamma} = 0$

In order now to exterminate the fluxions $\dot{u}$, $\dot{\alpha}$, $\dot{\beta}$, $\dot{\gamma}$, let theſe equations be reſpectively multiplied by 1, e, f, g, (yet unknown) and let all the products thence ariſing be added together; whence will be had $\overline{q + er + fs + gt} \times \dot{u} + \overline{q + er + fs + gt} \times \dot{\alpha} + \overline{q' + er' + fs' + gt'} \times \dot{\beta} + \overline{q'' + er'' + fs'' + gt''} \times \dot{\gamma} = 0$.

Make, now, $q + er + fs + gt = 0$,
$q' + er' + fs' + gt' = 0$,
$q'' + er'' + fs'' + gt'' = 0$,

From whence, there being as many equations as quantities, e, f, g, to be determined, the values of thoſe quantities will always be given from thence, in terms of the quantities $q, r, s, t, q', r', s', t'; q'', r'', s'', t''$ (excluſive of $\bar{q}$, $\bar{r}$, $\bar{s}$, $\bar{t}$,). Now, ſeeing all the terms of the equation $\overline{\bar{q} + e\bar{r} + f\bar{s} + g\bar{t}} \times \dot{u} + \overline{q + er + fs + gt} \times \dot{\alpha} + \overline{q' + er' + fs' + gt'} \times \dot{\beta} + \overline{q'' + er'' + fs'' + gt''} \times \dot{\gamma} = 0$, after the firſt $(\overline{\bar{q} + e\bar{r} + f\bar{s} + g\bar{t}} \times \dot{u})$ do thus actually vaniſh (by their coefficients being taken equal to nothing), it is evident, therefore, that $\bar{q} + e\bar{r} + f\bar{s} + g\bar{t}$ muſt alſo be $= 0$ (or *flux.* $\bar{\mathcal{Q}} + e$ *flux.* $\bar{R} + f \times$ *flux.* $\bar{S} + g \times$ *flux.* $\bar{\mathcal{T}} = 0$); where

e, f, g being quantities depending intirely upon q, r, s, t, &c. (excluſive of $\bar{q}$, $\bar{r}$, $\bar{s}$, $\bar{t}$), they muſt neceſſarily be invariable, or continue of the ſame value, let $\bar{q}$, $\bar{r}$, $\bar{s}$, $\bar{t}$, ſtand for which terms you will of the correſponding ſeries's, $q''' + q''''$ &c. $r''' + r''''$ &c. becauſe the quantities q, r, s, t; q', r', s', t', q'', r'', s'', t'', (on which e, f, g, depend) have themſelves a determinate value each, in the required circumſtance, when $Q + Q' + Q''$ &c. is a maximum, or minimum.

PROPOSITION.

Suppoſing y and u to be two flowing quantities, and that $\bar{Q}$, $\bar{R}$, $\bar{S}$, $\bar{T}$ &c. are quantities expreſſed in terms of y, u, and given coefficients; 'tis propoſed to find an equation, expreſſing the relation of y and u (or of $\bar{Q}$, $\bar{R}$, $\bar{S}$, $\bar{T}$, &c.) ſo that the fluent of $\bar{Q}\dot{y}$, correſponding to a given value of y, ſhall be a maximum *or* minimum, *and the fluents of $\bar{R}\dot{y}$, $\bar{S}\dot{y}$, $\bar{T}\dot{y}$, &c. all of them, at the ſame time, equal to given quantities.*

Let Q, Q', Q'', Q''', &c. be the different values of Q, that will ariſe, when y is, ſucceſſively, expounded by the terms of a given arithmetical progreſſion whoſe common difference is the indefinitely ſmall quantity y' (α, β, γ, δ, &c. denoting the reſpective values of u), and let R, R', R'', R''', &c. be the correſponding values of $\bar{R}$, &c. &c. Then it is well known that the ſum of all the quantities $Qy' + Q'y' + Q''y' + Q'''y' + Q''''y'$ &c. will be $=$ fluent of $\bar{Q}\dot{y}$; and the ſum of all the quantities $Ry' + R'y' + R''y' + R'''y' + R''''y'$ &c. $=$ fluent of $\bar{R}\dot{y}$, &c. But, *by the Lemma*, it appears, that $Q + Q' + Q'' + Q''' +$ &c. or $Qy' + Q'y' + Q''y' + Q'''y' +$ &c. (becauſe y' is conſtant) will be a *maximum* or *minimum*, and the quantities $Ry' + R'y' + R''y' + R'''y'$, &c. $Sy' + S'y' + S''y' + S'''y'$, &c. at the ſame time equal to given ones, when the relation of y and u (or of $\bar{Q}$, $\bar{R}$, $\bar{S}$, $\bar{T}$, &c.) is expreſſed by the equation, flux. $\bar{Q} + e \times$ flux. $\bar{R} + f \times$ flux. $\bar{S} + g \times$ flux. $\bar{T} = 0$: where e, f, g, &c. denote (unknown) conſtant quantities; and where, in taking the fluxions of $\bar{Q}$, $\bar{R}$, $\bar{S}$, $\bar{T}$, &c. the quantity u is, alone, to be conſidered as variable; becauſe the ſucceſſive values of y, entering reſpectively into Q, Q', Q'', Q''', &c. are conſtant quantities, being (by hypotheſis) ſuch as ſucceſſively ariſe from the terms of a given arithmetical progreſſion. Hence we have the following

GENERAL RULE

for the Reſolution of Iſoperimetrical Problems of all orders.

Take the fluxions of all the propoſed expreſſions (as well that reſpecting the maximum or minimum, as of the others whoſe fluents are to be given quantities, making that quantity, and likewiſe its fluxion, invariable, whereof the fluxion (as well as the quantity itſelf) enters into the ſaid expreſſions; and, having divided every-where by the fluxion of the other quantity made variable, let the quantities hence ariſing, joined to general coefficients 1, *e*, *f*, *g*, *&c. be united into one ſum, and the whole be made equal to nothing: from which equation (wherein the values of e, f, g, &c. may be either poſitive, or negative, or nothing, as the caſe requires), the required relation of the two variable quantities will be truly exhibited.*

Fig. 22.

To illuſtrate the uſe of the *rule* here laid down by an example, let x and y be ſuppoſed to repreſent the ordinate (PQ) and abſciſſa (AP) of a curve $ADQE$; and ſuppoſe $AFRG$ to be another curve, having the ſame abſciſſa, whoſe ordinate PR is, every-where, $= ax^m y^n$; 'tis required to find the relation of x and y, ſo that the area $BFGC$, anſwering to a given value of BC, ſhall be a *maximum* or *minimum*, at the ſame time that the correſponding area $BDEC$ is equal to a given quantity. Here, *by hypotheſis*, the fluent of $ax^m y^n \dot{y}$ is to be a *maximum* or *minimum*, and that of $x\dot{y}$ equal to a given quantity: taking, therefore, the fluxions of both expreſſions, &c. (making x alone variable, according to the *rule*), we thence get $max^{m-1} y^n \dot{y} - e\dot{y} = 0$: whence $x^{m-1} y^n = \frac{e}{ma}$; and conſequently $ax^m y^n \; (= PR) = \frac{ex}{m}$. Therefore, ſeeing PR is in a conſtant ratio to PQ, it is evident, that both the curves will be of the ſame kind; and that they will be both *hyperbolas*, or both *parabolas*, according as the values of the exponents $m - 1$, and n (in the general equation $x^{m-1}y^n = \frac{e}{ma}$) are *like*, or *unlike*, with regard to *poſitive* and *negative*. If $m - 1$ be *poſitive*, the equation gives a *minimum*; if *negative*, a *maximum*; but when $m - 1 = 0$, or when $m = 0$, the equation fails; in which caſes there will be neither a *maximum*, nor a *minimum*.

For another example, let the fluxions given be $\frac{y\dot{x}^3}{\dot{y}\dot{y}}$ and $\dot{x}$; the fluent of the former (anſwering to a given value of y) being to be a *minimum*, and that of the latter, at the ſame time, equal to a given quantity. Here ($\dot{x}$ being concerned independently, either, of its fluent x or fluxion $\ddot{x}$) let the fluxions of both expreſſions be taken, making $\dot{x}$ alone variable; whence, after dividing by $\ddot{x}$, we have $\frac{3y\dot{x}\dot{x}}{\dot{y}\dot{y}}$ and 1: therefore, in this caſe $\frac{3y\dot{x}\dot{x}}{\dot{y}\dot{y}} + e = 0$: whence $\dot{x} = a^{\frac{1}{2}} y^{-\frac{1}{2}} \dot{y}$ (ſuppoſing $a = -\frac{1}{3}e$); and conſequently $x = 2a^{\frac{1}{2}} y^{\frac{1}{2}}$; being an equation anſwering to the *common parabola*. The ſame concluſion may be otherwiſe derived (without ſecond-fluxions) by aſſuming $\frac{\dot{x}}{\dot{y}} = v$; whereby our two given expreſſions will be transformed to $y\dot{y}v^3$ and $\dot{y}v$: from whence, by the rule, we get $3v^2y\dot{y} + e\dot{y} = 0$; and therefore $v \left(\frac{\dot{x}}{\dot{y}}\right) = a^{\frac{1}{2}} y^{-\frac{1}{2}}$; whence $\dot{x} = a^{\frac{1}{2}} y^{-\frac{1}{2}} \dot{y}$, and conſequently $x = 2a^{\frac{1}{2}} y^{\frac{1}{2}}$, *the ſame as before.*

Fig. 23. If the *abſciſſa (AP)* of a curve *AQC* be denoted by x, and the *ordinate PQ* by y, and p be taken to expreſs the meaſure of the circumference of a circle whoſe diameter is unity; it is well known that the ſeveral fluxions, of the abſciſſa *AP*, curve-line *AQ*, area *APQ*, ſuperficies of the generated ſolid (by a rotation about the axis *AP*), and of the ſolid itſelf, will be, reſpectively, repreſented by $\dot{x}$, $\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$, $y\dot{x}$, $2py\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$, and $py^2\dot{x}$: if therefore, the fluxions of theſe different expreſſions be taken, as before (making $\dot{x}$ alone variable) we ſhall get $1 + \frac{e\dot{x}}{\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}} + fy + \frac{gy\dot{x}}{\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}} + hy^2 = 0$; being a general equation for determining the relation of x and y, when any one of thoſe five quantities (*viz.* the abſciſſa, curve-line, area, ſuperficies, or ſolid) is a *maximum* or *minimum*, and all,

or any number of the others, at the ſame time, equal to given quantities; wherein the coefficients e, f, g, and h, may be poſitive, negative, or nothing, as the caſe propoſed may require. Thus, for inſtance, if the length of the curve only be given, and the area correſponding is required to be a *maximum*, our equation will then become $\frac{e\dot{x}}{\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}} + fy = 0$, or $a^2\dot{x}^2 = y^2 \times \overline{\dot{x}\dot{x}+\dot{y}\dot{y}}$ (by making $a = -\frac{e}{f}$); whence $\dot{x} = \frac{y\dot{y}}{\sqrt{aa-yy}}$, and conſequently $x = a - \sqrt{aa-yy}$, or $2ax - x^2 = y^2$; anſwering to a circle; which figure is, therefore, more capacious than any other under equal bounds.

If, together with the ordinate (which, here, is always ſuppoſed given) the abſciſſa, at the end of the fluent, be given likewiſe, and the ſuperficies generated by the rotation of the curve about its axis be a *minimum*; then, from the ſame equation, we ſhall have $1 + \frac{gy\dot{x}}{\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}} = 0$, whence (making $a = -\frac{1}{g}$) $\dot{x}$ is found $= \frac{a\dot{y}}{\sqrt{yy-aa}}$, and from thence $x = a \times$ hyp. log. $\frac{y+\sqrt{yy-aa}}{a}$: which equation, by being impoſſible when y is leſs than a, ſhews that the curve (which is here the *catenaria)* cannot poſſibly meet the axis about which the ſolid is generated; and conſequently, that the caſe will not admit of any *minimum*, unleſs the firſt, or leaſt given value of y, exceeds a certain aſſignable magnitude.

When any, or all of the above-ſpecified quantities are given, and the contemporary fluent of ſome other expreſſion, as $\overline{\dot{x}\dot{x}+\dot{y}\dot{y}}|^{n} \times y^{m}\dot{y}^{1-2n}$, is required to be a *maximum* or *minimum*, our equation (by taking the fluxion of this laſt expreſſion, and joining it to the former) will then be $\overline{\dot{x}\dot{x}+\dot{y}\dot{y}}|^{n-1} \times 2n\dot{x}y^{m}\dot{y}^{1-2n} + d + \frac{e\dot{x}}{\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}} + fy + \frac{gy\dot{x}}{\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}} + hy^2 = 0$: which, when $m = 1$, and $n = -1$, will be *that* defining *the ſolid of the leaſt reſiſtance*; and this, when the length of the axis, only, is ſuppoſed to be given (without farther reſtrictions) will be expreſſed by $\overline{\dot{x}\dot{x}+\dot{y}\dot{y}}|^{-2} \times -2\dot{x}y\dot{y}^{3}+d=0$, or $2y\dot{y}^{3}\dot{x}=d\times\overline{\dot{x}\dot{x}+\dot{y}\dot{y}}|^{2}$; being the caſe firſt conſidered by Sir ISAAC NEWTON.—If both the length and the ſolid content be given, the equation will be $-2\dot{x}y\dot{y}^{3}\times\overline{\dot{x}\dot{x}+\dot{y}\dot{y}}|^{-2}+d+by^{2}=0$; but if, beſides theſe, the ſuperficies is given likewiſe, it will then be $-2\dot{x}y\dot{y}^{3}\times\overline{\dot{x}\dot{x}+\dot{y}\dot{y}}|^{-2}+d+\frac{gy\dot{x}}{\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}}+by^{2}=0$.

Thus, in like manner, by aſſuming $m=-\frac{1}{2}$, and $n=\frac{1}{2}$, we have $\frac{y^{-\frac{1}{2}}\dot{x}}{\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}}+d+\frac{e\dot{x}}{\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}}+fy+\frac{gy\dot{x}}{\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}}+by^{2}=0$; being the general equation of the *curve of the ſwifteſt deſcent*; which, when e, f, g, and b are all of them taken equal to nothing, will become $\frac{y^{-\frac{1}{2}}\dot{x}}{\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}}+d$; which is the caſe conſidered by many Others, anſwering to the cycloid. When the length of the arch deſcribed in the whole deſcent (along with the values of x and y) is given, the equation will then be $\frac{y^{-\frac{1}{2}}\dot{x}}{\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}}+d+\frac{e\dot{x}}{\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}}=0$, or $\overline{e+y^{-\frac{1}{2}}}|^{2}\times\dot{x}^{2}=d^{2}\times\overline{\dot{x}\dot{x}+\dot{y}\dot{y}}$. And thus may the relation of x and y be determined in any other caſe, and under any number of reſtrictions; provided that one of theſe quantities, only, enters into the ſeveral expreſſions given.—When both x and y are concerned, as well as their fluxions, the conſideration becomes more complicated; nor does it ſeem practicable to arrive at a General Rule, to anſwer equally in ſuch caſes. Nevertheleſs, if the ultimate values of x and y are ſuppoſed given, or the required curve is to paſs through two given points, without being confined to farther limitations, except that of the maximum or minimum (which caſe is the chief, and the moſt uſeful that can occur); then the method of ſolution may be as follows:

Take the fluxion of the given expreſſion (whoſe fluent is to be a maximum or minimum) making $\dot{x}$ *alone variable; and, having divided by* $\ddot{x}$, *let the quotient be denoted by* u.

Take, again, the fluxion of the ſame expreſſion, making x, *alone, variable; which divide by* $\dot{x}$: *then will this laſt quotient* $=u$.

From which equation the value of u, and the relation of x and y will be determined.

Thus, for example, if the expreſſion propounded (whoſe fluent, correſponding to any given values of x and y, is to be a *minimum)* were to be $\overline{f+gx} \times \frac{y\dot{x}^3}{\dot{y}\ddot{y}}$; then the fluxion thereof, when $\dot{x}$ alone is made variable, being $\overline{f+gx} \times \frac{3y\dot{x}^2\ddot{x}}{\dot{y}\ddot{y}}$, and, when x alone is made variable, equal to $\frac{gy\dot{x}^4}{\dot{y}\ddot{y}}$, we here have $\overline{f+gx} \times \frac{3y\dot{x}^2}{\dot{y}\ddot{y}} = u$, and $\frac{gy\dot{x}^3}{\dot{y}\ddot{y}} = \dot{u}$; the latter of which, divided by the former, gives $\frac{\dot{u}}{u} = \frac{\frac{1}{3}g\dot{x}}{f+gx}$; whence hyp. log. $u =$ hyp. log. $\overline{f+gx}|^{\frac{1}{3}} +$ hyp. log. d (d being any conſtant quantity). Conſequently $u = d \times \overline{f+gx}|^{\frac{1}{3}}$; which value being ſubſtituted in the equation $\overline{f+gx} \times \frac{3y\dot{x}^2}{\dot{y}\ddot{y}} = u$, we thence have $\overline{f+gx}|^{\frac{2}{3}} \times \dot{x}^2 = \frac{1}{3}dy^{-1}\dot{y}^2$, or $\overline{f+gx}|^{\frac{1}{3}} \times \dot{x} = cy^{-\frac{1}{2}}\dot{y}$ (making $c = \sqrt{\frac{d}{3}}$), and conſequently by taking the fluent again, we have $\frac{3 \times \overline{f+gx}|^{\frac{4}{3}} - 3f^{\frac{4}{3}}}{4g} = 2cy^{\frac{1}{2}}$; expreſſing the general relation of x and y, ſuppoſing them both to begin to be generated together.—If $f = 1$ and $g = 0$, the fluxion propounded will become $\frac{y\dot{x}^3}{\dot{y}\ddot{y}}$ (the ſame as in the firſt of the former examples); and here, $\dot{x}$ being $= cy^{-\frac{1}{2}}\dot{y}$, x will be $= 2cy^{\frac{1}{2}}$, anſwering (as before) to the common *parabola*.—But if $f = 0$ and $g = 1$, then our given fluxion will become $\frac{xy\dot{x}^3}{\dot{y}\ddot{y}}$, and the reſulting equation will be $\frac{3}{4}x^{\frac{4}{3}} = 2cy^{\frac{1}{2}}$, or $x^8 = a^5y^3$ (a being put $= \overline{\frac{8c}{3}}|^{\frac{6}{5}}$); which alſo anſwers to a *parabola*, but of an higher order.—The very ſame concluſions will, in like manner, be brought out by making $\dot{y}$ and y, ſucceſſively, variable (inſtead of $\dot{x}$ and x). For, here, the two fluxions reſulting (after having divided by $\ddot{y}$

and $\dot{y}$) appear to be $\frac{\overline{f+gx} \times -2yx^3}{\dot{y}^3} = u$, and $\frac{\overline{f+gx} \times \dot{x}^3}{\dot{y}^2} = \dot{u}$: whence, dividing the latter by the former, we have $\frac{-\frac{1}{2}\dot{y}}{y} = \frac{\dot{u}}{u}$; and therefore $-$ hyp. log. $y^{\frac{1}{2}}$ $+$ hyp. log. $a =$ hyp. log. u (a being any conſtant quantity). Conſequently, $ay^{-\frac{1}{2}} = u = \frac{\overline{f+gx} \times -2y\dot{x}^3}{\dot{y}^3}$, and $\overline{f+gx}|^{\frac{1}{3}} \times \dot{x} = cy^{-\frac{1}{2}}\dot{y}$ (c^3 being put $= -\frac{1}{2}a$). Hence, by taking the fluent again, we have $\frac{3 \times \overline{f+gx}|^{\frac{4}{3}} - 3f^{\frac{4}{3}}}{4g} = 2cy^{\frac{1}{2}}$, the very ſame as before.

Of

Of the REDUCTION of Algebraic Equations, by the Method of SURD DIVISORS; containing an Explanation of the Grounds of that Method, as it is laid down by Sir ISAAC NEWTON in his Univerſal Arithmetick.

THE reduction of equations by ſurd diviſors, which is looked upon, by many, as a very intricate kind of ſpeculation, is founded on the ſame principles with the method of extracting the roots of common quadratic equations, by compleating of the ſquare, with this difference only, that the ſquares on both ſides of the equation are, *here*, affected by the unknown quantity; whereas, in the common method, the ſquare on the right-hand ſide is a quantity intirely known. What we, therefore, have to do, is, *to ſeparate, and ſo order the terms of the equation given, that both ſides thereof may (if poſſible) be complete ſquares.*

CASE I. *If the given equation be a biquadratic one*, let it be $x^4+px^3+qx^2+rx+s=0$, and let there be aſſumed $\overline{xx+\frac{1}{2}px+\mathcal{Q}}|^2-\overline{Ax+B}|^2=x^4+px^3+qx^2+rx+s(=0)$; that is, let the values of the quantities $\mathcal{Q}$, A and B be ſuppoſed ſuch, that the coefficients of the powers of x, when $xx+\frac{1}{2}px+\mathcal{Q}$ and $Ax+B$ are ſquared, ſhall agree, or be the ſame, in every term, with thoſe of the equation given. Then, the ſaid quantities being actually ſquared, our equation will become

$$\left.\begin{array}{l} x^4+px^3+2\mathcal{Q}x^2 \quad * \quad\quad * \\ \qquad\qquad +\frac{1}{4}p^2x^2+p\mathcal{Q}x+\mathcal{Q}^2 \\ \qquad\qquad -AAx^2-2ABx-B^2 \end{array}\right\}=x^4+px^3+qx^2+rx+s.$$

From whence, by equating the coefficients of the homologous powers, and putting $a=q-\frac{1}{4}pp$, we have,

1. $2\mathcal{Q}+\frac{1}{4}p^2-A^2=q$, or, $2\mathcal{Q}=A^2+\alpha$;
2. $p\mathcal{Q}-2AB=r$, or, $p\mathcal{Q}=2AB+r$;
3. $\mathcal{Q}^2-B^2=s$, or, $\mathcal{Q}^2=B^2+s$.

Now, if the value of $\mathcal{Q}$, as given by the firſt equation, be ſubſtituted in the other two, we ſhall get $\frac{1}{2}pA^2 - 2AB = \beta$, and $\frac{1}{4}A^4 + \frac{1}{2}\alpha A^2 - B^2 = \zeta$, ſuppoſing $\beta = r - \frac{1}{2}\alpha p$, and $\zeta = s - \frac{1}{4}\alpha\alpha$. In which equations the unknown quantities appertaining to the latter of the two aſſumed ſquares are only concerned, and from which their values might be found. But as the reſulting equation, when one of the quantities is exterminated, riſes to the ſixth dimenſion, and would, perhaps, require more trouble to reduce it than, even, the original one propounded, little advantage would be reaped therefrom. Inſtead, therefore, of proceeding farther in a direct manner, it may be of uſe to try, whether ſome property, or relation of theſe quantities cannot from hence be diſcovered, whereby we may be enabled to gueſs at their values; which may be afterwards tried by means of the equations here exhibited.

Firſt, then, it is evident, if both A and B are either integers or rational quantities, that the equation $\overline{x^2 + \frac{1}{2}px + Q}|^2 - \overline{Ax + B}|^2$ $(= x^4 + px^3 + qx^2 + rx) = 0$ will, even after it is reduced to $x^2 + \frac{1}{2}px + Q = Ax + B$, be intirely free from radical quantities. In which caſe, the method of *rational diviſors* taking place, a reduction by means of ſurd quantities, or diviſors, as they do not naturally ariſe in the conſideration, cannot be of uſe. But the relation of the given quantities p, q, r, s (which we ſhall always, hereafter, conſider as integers) may be ſuch, that the values of A and B ſhall be radical quantities, commenſurate to each other; in which caſe, where the method of *rational diviſors* fails, we may aſſume $\sqrt{n}$ for the common radical diviſor, and expreſs the quantities themſelves by $k\sqrt{n}$, and $l\sqrt{n}$; that is, we may make $A = k\sqrt{n}$, and $B = l\sqrt{n}$; by which means our two equations, derived above, will be changed to $\frac{1}{2}pk^2n - 2kln = \beta$, and $\frac{1}{4}k^4n^2 + \frac{1}{2}\alpha k^2n - ln^2 = \zeta$, or to $\frac{1}{2}pk^2 - 2kl = \frac{\beta}{n}$, and $\frac{1}{2}k^4n + ak^2 - 2l^2 = \frac{2\zeta}{n}$, reſpectively.

Now, ſince n is ſuppoſed to be an integer, it is plain from hence (conſidering k and l alſo as integers, or the halves of ſuch) that $\frac{\beta}{n}$ and $\frac{2\zeta}{n}$ muſt be integers likewiſe, or, at leaſt, the halves of integers; and conſequently that n (whoſe value we are here ſeeking) ought to be ſome common integral diviſor of β and 2ζ.

Moreover, with regard to k and l, it is evident from the firſt of thoſe equations $\left(\frac{1}{2}pk - 2l = \frac{\frac{\beta}{n}}{k}\right)$ that the former (k) ought to be ſome diviſor of $\frac{\beta}{n}$; and that, if the quotient $\frac{\frac{\beta}{n}}{k}$ be taken from $\frac{1}{2}pk$, the remainder $\left(\frac{1}{2}pk - \frac{\frac{\beta}{n}}{k}\right)$ will be the double of l.

It farther appears, from the equations $Q = \frac{A^2 + \alpha}{2}$, and $Q^2 = B^2 + s$, by ſubſtituting for A^2 and B^2 their equals nk^2 and nl^2, that Q will be $= \frac{\alpha + nk^2}{2}$, and $l^2 = \frac{QQ - s}{n}$. From the former of which Q will be known, when n and k are known; by means whereof and the other equation, l may be, a ſecond time, found; and the agreement, or coincidence of this value with that before determined for l, will prove the ſolution in all reſpects; becauſe then the conditions of three original equations ($2Q = A^2 + \alpha$, $pQ = 2AB + r$, $Q^2 = B^2 + s$) will be all compleatly fulfilled.—It is true indeed, that no immediate regard, in the concluſion, is had to the ſecond of thoſe equations; but then it ought to be obſerved, that the equation $\frac{1}{2}pk - \frac{\frac{\beta}{n}}{k}$, whereby l is, the firſt time, found, is a conſequence thereof, being derived from *that*, and the firſt equation, conjunctly: and it is known, that, whatever values are diſcovered for unknown quantities, by means of equations derived from others, ſuch values do equally anſwer the conditions of the original equations propounded.

Seeing the method of ſolution, above traced out, depends upon the aſſuming proper diviſors of β, 2ζ, and $\frac{\beta}{n}$, for the values of n and k, it may therefore be expedient, firſt of all, in order to bring the work into leſs compaſs, to reject ſuch diviſors of thoſe quantities (if we can by any means diſcover them) which we know are not for our purpoſe. And this may, in ſome meaſure, be effected, from the conſideration of the properties of even and odd numbers.

In order to which $Q = \frac{\alpha + nk^2}{2}$ being previouſly transformed to nk^2 $(= 2Q - \alpha) = 2Q - q + \overline{\tfrac{1}{2}p}\vert^2 = \overline{\tfrac{1}{2}p}\vert^2 - f$ (by putting $q - 2Q = f$), it is evident, from thence, that *if p be an odd number*, $p^2 - 4f$, and conſequently its equal $4nk^2$, will likewiſe be an odd number; becauſe an even number $(4f)$ ſubtracted from the ſquare of an odd one, always, leaves *odd*. Therefore, ſeeing $4k^2 \times n$ is here an odd number, both n and $4k^2$ muſt be odd (for the product of two even numbers, or of an odd one and an even one, is even, and not odd). Whence it follows, becauſe $\overline{2k}\vert^2$ is odd, that $2k$ muſt be odd too; and conſequently k the half of an odd number.

Now, ſeeing p, n, and $2k$ are all of them odd numbers (when p is ſuch) they may, therefore, be expreſſed by $2a + 1$, $2b + 1$, and $2c + 1$, reſpectively; a, b, and c being integers: in conſequence of which aſſumption the equation $4nk^2 = p^2 - 4f$, will, by ſubſtitution, be changed to $8bc^2 + 8bc + 2b + 4c^2 + 4c + 1 = 4a^2 + 4a + 1 - 4f$, or $2bc^2 + 2bc + \frac{1}{2}b + c^2 + c = a^2 + a - f$. From whence it is manifeſt, as all the terms, but $\frac{1}{2}b$, are known to be integers, that $\frac{1}{2}b$ muſt be an integer likewiſe: and ſo, b being an even number, it follows that n, or $2b + 1$, muſt be the double of an even number (or a multiple of 4) increaſed by unity. Therefore all the diviſors of β and 2ζ that have not this property may be ſafely rejected, as not for the purpoſe.

In like manner, *if p be even*, the ſame limitations will take place, *provided that r is odd*; which will be the caſe when Q is the half of an odd number (For, when Q is an integer, $A^2 = \overline{\tfrac{1}{2}p}\vert^2 - f$) and B^2 $(= Q^2 - s)$ being integers, their product A^2B^2 will be an integer, and conſequently the ſquare root thereof AB (being rational) will likewiſe be an integer;

and ſo, pQ and $2AB$ being both even numbers, their difference r, as given by the equation $pQ = 2AB + r$, would be even, and not odd). Therefore, ſeeing B^2, or its equal nl^2, is here equal to the ſquare of half of an odd number (Q) joined to an integer ($-s$), in the ſame manner as nk^2 was in the preceding caſe; it is evident, from the reaſoning there laid down, that the value of n is ſubject to the very ſame reſtrictions here, as there.—Other limitations might be pointed out, from the properties of even and odd numbers, were the thing worth purſuing farther. What is already delivered on this head is ſufficient for the purpoſe, and for the underſtanding of Sir ISAAC NEWTON: I ſhall therefore, from the ſeveral concluſions above derived, now lay down the ſubſequent

RULE

for the reduction of an equation ($x^4 + px^3 + qx^2 + rx + s = 0$) *of four dimenſions.*

Make $\alpha = q - \frac{1}{4}p^2$, $\beta = r - \frac{1}{2}\alpha p$, and $\zeta = s - \frac{1}{4}\alpha\alpha$; *then put for n ſome common integral diviſor of* β *and* 2ζ, *that is neither a ſquare, nor diviſible by a ſquare, and which being divided by* 4, *ſhall leave unity, if either p or r be odd. Put alſo for k ſome diviſor of* $\frac{\beta}{n}$, *if p be even, or half of the odd diviſor if p be odd: take the quotient from* $\frac{1}{2}pk$, *and call half the remainder l. Make* $Q = \frac{\alpha + nkk}{2}$, *and try if n divides* $QQ - s$, *and the root of the quotient be equal to l; if it ſo happen, then the propoſed equation, by means of the values thus determined, will be reduced to* $xx + \frac{1}{2}px + Q = \pm\sqrt{n} \times \overline{kx + l}$.

That the diviſor n ought not here to be a ſquare, is evident from what has been already remarked, ſince both A and B would then be rational quantities; and that *it* ought not to be diviſible by a ſquare, will alſo appear, if it be conſidered that k and l in the equations $k\sqrt{n} = A$, and $l\sqrt{n} = B$, are to be taken the greateſt, and n the leaſt, that the caſe will admit of.

No regard in this Rule is had to that circumſtance, in which β happens to be nothing. Sir ISAAC NEWTON here directs, *to take k alſo equal to nothing.* The reaſon of which depends on the equation $\frac{1}{2}pk^2 - 2kl = \frac{\beta}{n}$, which in this caſe becomes $\frac{1}{2}pk^2 - 2kl = 0$; where one root, or value of k muſt, neceſſarily, be nothing. Therefore Q being $= \frac{1}{2}\alpha$, we have $l\sqrt{n}\ (= \sqrt{Q^2 - s}) = \sqrt{\frac{1}{4}\alpha^2 - s}$; ſo that, by a direct proceſs, our given equation is here reduced to $x^2 + \frac{1}{2}px + \frac{1}{2}\alpha = \sqrt{\frac{1}{4}\alpha\alpha - s}$, wherein α is given $= q - \frac{1}{4}pp$.

The celebrated mathematician MACLAURIN, who, in his *Treatiſe of Algebra*, has commented largely on the diſcoveries of our Author, ſeems to repreſent this part of the General Rule, as not well grounded; laying down, at the ſame time, two new Rules, in order to ſupply the defect. Which Rules, I muſt confeſs, to me appear unneceſſary; ſince it is certain, that the method of ſolution, as laid down by Sir ISAAC NEWTON, is more direct and eligible in this particular caſe than in any other. It muſt be allowed, indeed, that the manner of applying the Rule, in this caſe, is left ſomewhat obſcure; but as to his directing, to take $k = 0$, when $\beta = 0$, it cannot, I am fully perſuaded, admit of any well-grounded objection. For, though "it does not neceſſarily follow that k muſt be $= 0$, when $\beta = 0$," yet the taking of k thus $= 0$, involves no abſurdity; ſeeing one value of k (at leaſt) will be nothing.—The truth is, there are three different values that k may admit of (as appears by the ſubſequent note*); all of which will, equally, fulfil the ſeveral conditions required, and bring out the very ſame concluſion. Thus the value of k, in the equation

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* If the ſquare of half the ſecond of the original equations, $2Q - \alpha = AA$, $pQ - r = 2AB$, $QQ - s = BB$, be ſubtracted from the product of the other two, there will be obtained the equation $Q^3 - \frac{1}{2}qQ^2 + \overline{\frac{1}{4}pr - s} \times Q + \frac{1}{2} \times \alpha s - \frac{1}{4}rr = 0$; wherein the unknown quantity Q is alone concerned; which equation being of three dimenſions, the root Q, and conſequently k $\left(\frac{2Q - \alpha}{n}\right)$ will admit of three different values.—From this equation it alſo appears, that Q muſt always be a diviſor of the quantity $\alpha s - \frac{1}{4}rr$; which is a circumſtance taken notice of by our Author.

$x^4+2x^3-37x^2-38x+1=0$ (propoſed by this gentleman) may be 0, 3 or 4; or, which comes to the ſame, the equation itſelf may be reduced to $x^2+x-19=\pm 6\sqrt{10}$, $x^2+x+\frac{7}{2} = \pm\sqrt{5}\times\overline{3x+\frac{3}{2}}$, or to $x^2+x-3=\pm\sqrt{2}\times\overline{4x+2}$. All which are, in effect, but one and the ſame equation, as will appear by ſquaring both ſides of each, and properly tranſpoſing; from whence the given equation $x^4+2x^3-37x^2-38x+1=0$, will in every caſe emerge. The ſecond of theſe equations is that brought out by Mr. MACLAURIN; but the firſt, which is that found by our Author's Rule, is not only more commodious, but eaſier to be determined, being derived by a direct, and very ſhort proceſs.—And ſo much for equations of four dimenſions.

CASE. II. *If the equation to be reduced is of ſix dimenſions*, let it be $x^6+px^5+qx^4+rx^3+sx^2+tx+v=0$; and let there be aſſumed $\overline{x^3+\frac{1}{2}px^2+Qx+R}|^2-\overline{Ax^2+Bx+C}|^2 = x^6+px^5+qx^4+rx^3+sx^2+tx+v\ (=0)$; which, by involution and tranſpoſition, will give

$$\left.\begin{array}{llllll}
+2Qx^4 & +2Rx^3 & & & & \\
+\frac{1}{4}p^2x^4 & +pQx^3 & +pRx^2 & & & \\
& & +Q^2x^2 & +2QRx & +R^2 & \\
-qx^4 & -rx^3 & -sx^2 & -tx & -v &
\end{array}\right\} = \begin{array}{l} A^2x^4+2ABx^3+ \\ 2ACx^2+B^2x^2+ \\ 2BCx+C^2. \end{array}$$

From whence, by equating the coefficients of the homologous powers, and writing $\alpha=q-\frac{1}{4}pp$, we have,

1. $2Q-\alpha=A^2$;
2. $2R+pQ-r=2AB$;
3. $pR+Q^2-s=2AC+B^2$;
4. $2QR-t=2BC$;
5. $R^2-v=C^2$.

If now the value of $Q\ (=\frac{1}{2}A^2+\frac{1}{2}\alpha)$ as given by the firſt of theſe equations, be ſubſtituted for Q, in the ſecond, we ſhall get $2R+\frac{1}{2}pA^2+\frac{1}{2}p\alpha-r=2AB$; and conſequently $R=AB-\frac{1}{4}pA^2+\frac{1}{2}\beta$ (by putting $\beta=r-\frac{1}{2}p\alpha$): which value, together with that of Q, being ſubſtituted in the three remaining equations, we ſhall have,

$$1.\ pAB - \tfrac{1}{4}p^2A^2 + \tfrac{1}{2}p\beta + \tfrac{1}{4}A^4 + \tfrac{1}{2}\alpha A^2 + \tfrac{1}{4}\alpha^2 - s = 2AC + B^2,$$
$$2.\ A^3B - \tfrac{1}{4}pA^4 + \tfrac{1}{2}\beta A^2 + \alpha AB - \tfrac{1}{4}p\alpha A^2 + \tfrac{1}{2}\alpha\beta - t = 2BC,$$
$$3.\ A^2B^2 - \tfrac{1}{2}pA^3B + \beta AB + \tfrac{1}{16}p^2A^4 - \tfrac{1}{4}p\beta A^2 + \tfrac{1}{4}\beta^2 - v = C^2:$$

which, by putting $\gamma = s - \frac{1}{2}p\beta$, $\zeta = \gamma - \frac{1}{4}\alpha\alpha$, $\eta = t - \frac{1}{2}\alpha\beta$, and $\theta = v - \frac{1}{4}\beta\beta$, will be reduced to

$$pAB - \tfrac{1}{4}p^2A^2 + \tfrac{1}{4}A^4 + \tfrac{1}{2}\alpha A^2 - \zeta = 2AC + B^2,$$
$$A^3B - \tfrac{1}{4}pA^4 + \tfrac{1}{2}\beta A^2 + \alpha AB - \tfrac{1}{4}p\alpha A^2 - \eta = 2BC, \text{ and}$$
$$A^2B^2 - \tfrac{1}{2}pA^3B + \beta AB + \tfrac{1}{16}p^2A^4 - \tfrac{1}{4}p\beta A^2 - \theta = C^2, \text{ reſpectively.}$$

Now, if the values of A, B and C are ſuch, as to admit of ſome common ſurd-diviſor, let that diviſor *(as in the preceding Caſe)* be denoted by $\sqrt{n}$, and the quantities themſelves by $k\sqrt{n}$, $l\sqrt{n}$, and $m\sqrt{n}$, reſpectively: then, ſubſtitution being made and every equation divided by n, we ſhall have,

$$1.\ pkl - \tfrac{1}{4}p^2k^2 + \tfrac{1}{4}nk^4 + \tfrac{1}{2}\alpha k^2 - \frac{\zeta}{n} = 2km + l^2,$$
$$2.\ nk^3l - \tfrac{1}{2}pnk^4 + \tfrac{1}{2}\beta k^2 + \alpha kl - \tfrac{1}{4}p\alpha k^2 - \frac{\eta}{n} = 2lm,$$
$$3.\ nk^2l^2 - \tfrac{1}{2}pnk^3l + \beta kl + \tfrac{1}{16}p^2nk^4 - \tfrac{1}{4}p\beta k^2 - \frac{\theta}{n} = m^2.$$

From whence it appears (ſince k, l, and m are here conſidered as integers, or as the halves of ſuch) that ζ, η, and θ ought to be all of them diviſible by n; or, which is the ſame, that n ought to be ſome common integral diviſor of the quantities ζ, η, and θ *.

Furthermore, with reſpect to the limitations to which k is ſubject, let the ſeveral terms in the former part of the firſt of our three equations, in which k is found (in order to abbreviate the work) be denoted by Fk; then will the equation itſelf be changed to $Fk - \frac{\zeta}{n} = 2km + l^2$. And, in the very ſame manner, our other two equations will be changed to $Gk - \frac{\eta}{n} = 2lm$, and $Hk - \frac{\theta}{n} = m^2$, reſpectively.

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* Sir ISAAC NEWTON directs to take n, ſome common diviſor of 2ζ, η, and 2θ (inſtead of ζ, η, and θ); but this makes no difference, becauſe all diviſors of ζ and θ are alſo diviſors of 2ζ and 2θ; nor are there any diviſors of 2ζ and 2θ, but what will likewiſe be diviſors of ζ and θ, if we (as Sir ISAAC has done) admit into the conſideration ſuch fractions as have the powers 2 for their denominator; which ariſe from the value of $\frac{1}{2}p$, in the aſſumed equation, being a fraction of this kind, when p is an odd number.

Let now the ſquare of half the ſecond of theſe equations be ſubtracted from the product of the firſt and third, then will $FHk^2 - \frac{Fk\theta}{n} - \frac{Hk\zeta}{n} + \frac{\zeta\theta}{nn} - \frac{1}{4}G^2k^2 + \frac{Gk\eta}{2n} - \frac{\frac{1}{4}\eta^2}{nn} = 2km^3$; which, by dividing the whole by $2k$, and putting $\lambda = \zeta\theta - \frac{1}{4}\eta^2$, will, at length, become

$$\tfrac{1}{2}FHk - \tfrac{1}{8}G^2k - \tfrac{1}{2}H \times \frac{\zeta}{n} + \tfrac{1}{4}G \times \frac{\eta}{n} - \tfrac{1}{2}F \times \frac{\theta}{n} + \frac{\frac{\lambda}{2nn}}{k} = m^3:$$

where ζ, η, and θ being all diviſible by their common diviſor n (as is ſhewn above) it is manifeſt that $\frac{\lambda}{2nn}$ (in order that m may be an integer, or the half of an integer) ought alſo to be diviſible by its diviſor k, that is, k ought to be ſome diviſor of the quantity $\frac{\lambda}{2nn}$.

Again, with regard to l, let the value of R $(= \mathrm{AB} - \frac{1}{2}p\mathrm{Q} + \frac{1}{2}r)$, as given by the ſecond of the five original equations, be ſubſtituted in the fourth; by which means we have $2\mathrm{QAB} = p\mathrm{Q}^2 + r\mathrm{Q} - t = 2\mathrm{BC}$, or $2\mathrm{Q}nkl - p\mathrm{Q}^2 + r\mathrm{Q} - t = 2nlm$ (becauſe $\mathrm{A} = k\sqrt{n}$, $\mathrm{B} = l\sqrt{n}$, $\mathrm{C} = m\sqrt{n}$), and conſequently $\frac{r\mathrm{Q} - p\mathrm{Q}^2 - t}{nl} = 2m - 2k\mathrm{Q}$; where $2m - 2k\mathrm{Q}$ being an integer, it is evident that $\frac{r\mathrm{Q} - p\mathrm{Q}^2 - t}{nl}$ muſt be an integer alſo, and, conſequently, l ſome diviſor of the quantity $\frac{r\mathrm{Q} - p\mathrm{Q}^2 - t}{n}$. From whence l being found in numbers, the value of R $(= \mathrm{AB} - \frac{1}{2}p\mathrm{Q} + \frac{1}{2}r = nkl - \frac{1}{2}p\mathrm{Q} + \frac{1}{2}r)$ will be had likewiſe; and then, by means of the three laſt of the five original equations, the value of m may be alſo found, three ſeveral ways, and the truth of the ſolution thereby confirmed: for theſe equations, by ſubſtituting for A, B, and C, their equals $k\sqrt{n}$, $l\sqrt{n}$, and $m\sqrt{n}$, do become $\mathrm{R}^2 - v = nm^2$, $2\mathrm{QR} - t = 2nlm$, and $p\mathrm{R} + \mathrm{Q}^2 - s = 2nkm + nll$; from the firſt of which

m

$m = \sqrt{\frac{R^2 - v}{n}}$; from the ſecond, $m = \frac{QR - \frac{1}{2}t}{nl}$; and from the third, $m = \frac{QQ + pR - nll - s}{2nk}$: which values, therefore, when Q, R, *&c.* are rightly aſſumed, will be all found equal among themſelves; and our given equation, $x^6 + px^5 + qx^4 + rx^3 + sx^2 + tx + v = 0$, or $\overline{x^3 + \frac{1}{2}px^2 + Qx + R}|^2 - \overline{Ax^2 + Bx + C}|^2 = 0$, will then be reduced to $x^3 + \frac{1}{2}px^2 + Qx + R$ $(= \pm \overline{Ax^2 + Bx + C}) = \pm \sqrt{n} \times \overline{kx^2 + lx + m}$.

As to the limitations in the diviſors to be tried, with reſpect to even and odd numbers, the reaſoning thereon is the very ſame as in the preceding Caſe; which, therefore, it will be unneceſſary to repeat.—One circumſtance there is, indeed, that merits a particular regard, and that is, when $\lambda = 0$; in which caſe k (or one value of k at leaſt) will alſo be $= 0$, and the reduction will be performed by a direct proceſs. For, k being nothing, the three equations wherein k, l, and m are firſt introduced, will become $-\frac{\zeta}{n} = l^2$, $-\frac{\eta}{n} = 2lm$, and $-\frac{\theta}{n} = m^2$; whence $l\sqrt{n} = \sqrt{-\zeta}$, $m\sqrt{n} = \sqrt{-\theta}$, and conſequently λ $(= \zeta\theta - \frac{1}{4}\eta^2) = 0$, as it ought to be. Therefore, by ſubſtituting theſe values, and writing alſo inſtead of Q and R their equals $\frac{1}{2}\alpha$ and $\frac{1}{2}\beta$, the equation given, is here reduced to $x^3 + \frac{1}{2}px^2 + \frac{1}{2}\alpha x + \frac{1}{2}\beta = \pm x\sqrt{-\zeta} \pm \sqrt{-\theta}$.

CASE III. *If the equation is of eight dimenſions,* let it be $x^8 + px^7 + qx^6 + rx^5 + sx^4 + tx^3 + vx^2 + wx + z = 0$: Then, by aſſuming $\overline{x^4 + \frac{1}{2}px^3 + Qx^2 + Rx + S}|^2 - \overline{Ax^3 + Bx^2 + Cx + D}|^2 = x^8 + px^7 + qx^6 + rx^5 + sx^4 + tx^3 + vx^2 + wx + z$ $(= 0)$ and proceeding as in the two former Caſes, we ſhall have here,

1. $2Q - \alpha \; . \; . \; . \; . \; . \; . \; . \; . = A^2$,
2. $2R + pQ - r \; . \; . \; . \; . = 2AB$,
3. $2S + pR + QQ - s = 2AC + BB$,
4. $pS + 2QR - t \; . \; . \; . = 2AD + 2BC$,
5. $2QS + RR - v \; . \; . = 2BD + CC$,

6.

6. $2RS - w = 2CD$,
7. $SS \;\; - z = DD$.

Put now (as before) $A = k\sqrt{n}$, $B = l\sqrt{n}$, $C = m\sqrt{n}$, $D = n'\sqrt{n}$; put alſo (to ſhorten the work) $Q = Q'n + \frac{1}{2}\alpha$, $R = R'n + \frac{1}{2}\beta$, $S = S'n + \frac{1}{2}\gamma$; that is, let the quotients of Q, R, and S, when divided by the common diviſor n, be Q', R', and S', and the remainders $\frac{1}{2}\alpha$, $\frac{1}{2}\beta$, and $\frac{1}{2}\gamma$, reſpectively: then, to determine theſe laſt, which muſt be firſt known, before n can be known, let ſubſtitution be made in the ſecond and third equations, every-where diſregarding ſuch terms wherein n and its powers are involved. Thus, ſubſtitution being made in the ſecond equation, we have $2k'n + \beta + pQ'n + \frac{1}{2}p\alpha - r = 2kln$; where the homologous terms, in which n enters not, are β, $\frac{1}{2}p\alpha$, and $-r$: the others, therefore, being here diſregarded, we have $\beta + \frac{1}{2}p\alpha - r = 0$, or $\beta = r - \frac{1}{2}p\alpha$. In the very ſame manner, from the third equation, $\gamma + \frac{1}{2}p\beta + \frac{1}{4}\alpha\alpha - s = 0$, and conſequently $\gamma = s - \frac{1}{2}p\beta - \frac{1}{4}\alpha\alpha$.

Let ſubſtitution be now made in the fourth, fifth, ſixth, and ſeventh equations (ſtill diſregarding all ſuch terms as would involve the powers of n), and there will come out,

1. $\frac{1}{2}p\gamma + \frac{1}{2}\alpha\beta - t =$,
2. $\frac{1}{2}\alpha\gamma + \frac{1}{4}\beta\beta - v =$,
3. $\frac{1}{2}\beta\gamma - w \quad =$,
4. $\frac{1}{4}\gamma\gamma - z \quad =$.

Now, as all the other terms, that would ariſe in theſe equations (beſides thoſe put down) are affected with n, and are therefore diviſible thereby, it is manifeſt that the four quantities $\frac{1}{2}p\gamma + \frac{1}{2}\alpha\beta - t$, $\frac{1}{2}\alpha\gamma + \frac{1}{4}\beta\beta - v$, $\frac{1}{2}\beta\gamma - w$, and $\frac{1}{4}\gamma\gamma - z$, here brought out, muſt likewiſe be, all of them, diviſible by the ſame common diviſor n, when the equation given is capable of being reduced. If, therefore, no ſuch common diviſor (under the reſtrictions ſpecified in the preceding Caſes, depending on p, r, t, or w being an odd number) can be diſcovered (which will moſt commonly happen) the work will then be at an end.

From the ſame method of operation, which may be looked upon as a ſort of examination, whether the equation be reducible or not, we may find all the quantities to which n ought to be a common diviſor, when the equation given is of 10, 12, or a greater number of dimenſions.

Thus, let there be given $x^{2e}+px^{2e-1}+qx^{2e-2}+rx^{2e-3}+sx^{2e-4}+tx^{2e-5}$ &c. $=0$, and let there be aſſumed

$$\overline{x^{e}+\tfrac{1}{2}px^{e-1}+\overline{Q'n+\tfrac{1}{2}\alpha}\times x^{e-2}+\overline{R'n+\tfrac{1}{2}\beta}\times x^{e-3}+\overline{S'n+\tfrac{1}{2}\gamma}\times x^{e-4}\,\&c.}\,\Big|^{2}$$
$$-\,n\times\overline{kx^{e-1}+lx^{e-2}+mx^{e-3}\,\&c.}\,\Big|^{2}=x^{2e}+px^{2e-1}+qx^{2e-2}\,\&c.$$

then, by ſquaring $x^{e}+\frac{1}{2}px^{e-1}+\overline{Q'n+\frac{1}{2}\alpha}\times x^{e-2}$ &c. and tranſpoſing $x^{2e}+px^{2e-1}+qx^{2e-2}$ &c. it will appear, that the terms of this equation, in which n enters not, will be

$$\left.\begin{matrix}\alpha\\ \frac{1}{4}p^{2}\\ -q\end{matrix}\right\}\times x^{2e-2}+\left.\begin{matrix}\beta\\ \frac{1}{2}p\alpha\\ -r\end{matrix}\right\}\times x^{2e-3}+\left.\begin{matrix}\gamma\\ \frac{1}{2}p\beta\\ \frac{1}{4}\alpha^{2}\\ -s\end{matrix}\right\}\times x^{2e-4}+\left.\begin{matrix}\delta\\ \frac{1}{2}p\gamma\\ \frac{1}{2}\alpha\beta\\ -t\end{matrix}\right\}\times x^{2e-5}+\left.\begin{matrix}\varepsilon\\ \frac{1}{2}p\delta\\ \frac{1}{2}\alpha\gamma\\ \frac{1}{4}\beta\beta\\ -v\end{matrix}\right\}\times x^{2e-6}$$

&c. From the former half of which terms, all the quantities α, β, γ, &c. will be determined, by aſſuming the coefficients equal to nothing: thus we have $\alpha=q-\frac{1}{4}pp$, $\beta=r-\frac{1}{2}p\alpha$, $\gamma=s-\frac{1}{2}p\beta-\frac{1}{4}\alpha\alpha$, $\delta=t-\frac{1}{2}p\gamma-\frac{1}{2}\alpha\beta$, &c. And then, theſe quantities being known, the coefficients of the remaining terms will likewiſe be known; which ought, all of them, to be diviſible by n, in order that the reduction may ſucceed; that is, they ought to be ſuch, as to admit of a common diviſor (n) under the reſtrictions before ſpecified.

For example, if the equation given were to be of twelve dimenſions, as $x^{12}+px^{11}+qx^{10}+rx^{9}+sx^{8}+tx^{7}+vx^{6}+ax^{5}+bx^{4}+cx^{3}+dx^{2}+\acute{e}x+f=0$, we ſhould have $\alpha=q-\frac{1}{4}pp$, $\beta=r-\frac{1}{2}p\alpha$, $\gamma=s-\frac{1}{2}p\beta-\frac{1}{4}\alpha\alpha$, $\delta=t-\frac{1}{2}p\gamma-\frac{1}{2}\alpha\beta$, and $\varepsilon=v-\frac{1}{2}p\delta-\frac{1}{2}\alpha\gamma-\frac{1}{4}\beta\beta$; and the coefficients of the other ſix terms (whereof n ought to be a common diviſor) would be $\frac{1}{2}p\varepsilon+\frac{1}{2}\alpha\delta+\frac{1}{2}\beta\gamma-a$, $\frac{1}{2}\alpha\varepsilon+\frac{1}{2}\beta\delta+\frac{1}{4}\gamma\gamma-b$, $\frac{1}{2}\beta\varepsilon+\frac{1}{2}\gamma\delta-c$, $\frac{1}{2}\gamma\varepsilon+\frac{1}{4}\delta\delta-d$, $\frac{1}{2}\delta\varepsilon-\acute{e}$, and $\frac{1}{4}\varepsilon\varepsilon-f$.

Theſe operations, for finding of n, as this ſort of reduction is ſeldom poſſible in high equations, will moſt commonly end the work. If ſuch a value, however, ſhould be found for n, as to anſwer all the conditions above ſpecified, it is not by purſuing the ſame method of diviſors, laid down in the reſolution of the preceding Caſes, that the values of *k*, *l*, *&c.* can from thence be determined, without a prodigious deal of trouble. There are indeed various other means of trying theſe quantities, by aſſuming ſome of them, and finding the others from thence; and ſo proceeding on, changing the values continually, till all the conditions of the ſeveral equations, ariſing from the compariſon of the homologous terms, are fulfilled. But as this is exceedingly laborious, and ſeeing after all, the uſe of ſo great reductions (as the ſagacious Author himſelf obſerves) is very little, there not being, perhaps, one caſe in a thouſand in which they can ſucceed; I ſhall, therefore, deſiſt here.

THE
RESOLUTION
OF
Some GENERAL PROBLEMS in MECHANICS, and PHYSICAL ASTRONOMY.

PROBLEM I.

Fig. 24. *Suppoſe a ſyſtem of bodies* A, B, C *(connected together) to revolve about a center, or axis* (P), *with a given angular celerity; it is propoſed to find the momentum (k) which, acting at a given diſtance* QP *from the center, ſhall be juſt ſufficient to ſtop, or take away the whole motion of the ſyſtem.*

IF the given angular celerity of the ſyſtem, at any diſtance PG from the axis, be denoted by v, the celerities of the ſeveral bodies A, B, and C will be truly expreſſed by $\frac{AP}{PG}\times v$, $\frac{BP}{PG}\times v$, and $\frac{CP}{PG}\times v$, reſpectively. Hence (by the property of the *Lever*) it will be, as PQ is to AP, ſo is $\left(\frac{AP}{PG}\times v\times A\right)$ the momentum of the body A, to $\frac{AP^2}{QP\times PG}\times v\times A$, the momentum, which acting at Q, is a juſt counterpoiſe to the action of A. And, in the very ſame manner, the momentum, acting at Q, ſufficient to take away the motion of B, appears to be $\frac{BP^2}{QP\times PG}\times v\times B$; and ſo on. Whence it is manifeſt, that the ſum of all *theſe*, muſt be the true momentum required; or that $k = v\times\frac{A\times AP^2 + B\times BP^2 + C\times CP^2}{PG\times QP}$. *Q. E. I.*

COROLLARY I.

If the motion of the ſyſtem is that which might be produced by any given momentums a, b, c (or forces capable of producing thoſe momentums) acting on the bodies A, B, C, in directions perpendicular to AP, BP, and CP; then (by the property of the lever) the force $a\times\frac{AP}{QP}$, acting at Q, having the ſame effect to turn the ſyſtem about its axis, as the force *a*, acting at the diſtance AP, *&c.* it follows that the force, which, by acting at Q, is ſufficient to deſtroy the whole motion of the ſyſtem, will here be $a\times\frac{AP}{QP}+b\times\frac{BP}{QP}+c\times\frac{CP}{QP}$: which being ſubſtituted in the room of *k*, our general equation, in the laſt article, will become $v=PG\times\frac{a\times AP+b\times BP+c\times CP}{A\times AP^2+B\times BP^2+C\times CP^2}$; ſhewing the angular celerity at the diſtance PG, produced in the ſyſtem by the action of the given forces; which celerity is, therefore, in proportion to the celerity $\left(\frac{a}{A}\right)$ that the given force (or momentum) *a* is capable of producing in the ſingle body A, as $\frac{A\times PG}{a}\times\frac{a\times AP+b\times BP+c\times CP}{A\times AP^2+B\times BP^2+C\times CP^2}$ to unity.

COROLLARY II.

If the momentum *k* be given equal to *that* of the whole ſyſtem (A, B, C) in a direction perpendicular to the line PGQ paſſing through the common center of gravity G; then the length of the lever (PQ) by which *k* acts, may be determined from hence. For, the celerity of the point G being repreſented by *v*, the momentum laſt named will, by the property of the center of gravity, be rightly defined by $v\times\overline{A+B+C}$; which being ſubſtituted in the room of *k*, we thence get $QP=\frac{A\times AP^2+B\times BP^2+C\times CP^2}{\overline{A+B+C}\times GP}$; exhibiting the diſtance of the *center of percuſſion* (Q) at which an immovable obſtacle receives the whole force of the ſtroke.

COROLLARY III.

If a ſingle body S, equal to the ſum of all the bodies A, B, C, be ſuppoſed to revolve (independent of the others) about the ſame center, with the common angular celerity of the ſyſtem, its momentum $\frac{SP}{PG} \times v \times S$, or $\frac{SP}{PG} \times v \times \overline{A + B + C}$, will be in proportion to the momentum *k* (given by the Propoſition) as QP × SP to $\frac{A \times AP^2 + B \times BP^2 + C \times CP^2}{A + B + C}$. By making theſe two quantities equal to each other, we have $SP = \frac{A \times AP^2 + B \times BP^2 + C \times CP^2}{\overline{A + B + C} \times QP}$, for the diſtance of the body S from the axis of motion, when its momentum is equal to the momentum *k*, or when equal forces, applied to the ſingle body at S, and to the ſyſtem at Q, can take away, or produce equal angular celerities in both, about the common axis of motion P.

COROLLARY IV.

Hence, if the point Q be ſuppoſed to coincide with S, our laſt equation will become $SP = \sqrt{\frac{A \times AP^2 + B \times BP^2 + C \times CP^2}{A + B + C}}$, ſhewing the diſtance of the *center of gyration*, or the place of the body S, where the ſame force can take away, or produce the whole motion of the ſyſtem A, B, C, as can take away, or produce the motion of the ſingle body S, equal to the ſum of all the former, and revolving with the ſame angular celerity.

COROLLARY V.

But if the point Q be taken in the common center of gravity G of the ſyſtem, and G*g* and S*s* be drawn perpendicular to the horizontal line TP; then, the force of gravity by which the whole ſyſtem is urged in the direction G*g* perpendicular to the horizon, being the ſum of all the weights (A + B + C) it is plain that the part of it acting in a direction perpendicular to PS, whereby the motion about the center is accelerated, will be $\overline{A + B + C} \times \frac{Pg}{PG}$. But the force whereby the weight S, in a direction perpendicular to the ſame PS, is accelerated, is equal to $S \times \frac{Ps}{PS} = \overline{A + B + C} \times \frac{Pg}{PG}$ (becauſe $S = A + B + C$, and $\frac{Ps}{PS} = \frac{Pg}{PG}$). Therefore, ſeeing the forces acting at S and Q are here equal, it is evident, from *Corol. III.* that the diſtance SP, ſo that the ſame angular celerity may be produced in the ſingle body as in the ſyſtem, will be truly exhibited by the general equation $SP = \frac{A\times AP^2 + B\times BP^2 + C\times CP^2}{A + B + C\times QP}$ there derived; the point S thus determined being the center of oſcillation, and the ſame with the center of percuſſion, found in *Cor. II.* having its diſtance from the axis, equal to a third proportional to the diſtance of the center of gravity and that of gyration, determined in *Corol. IV.*

COROLLARY VI.

Hence, alſo, the preſſure on the axis of ſuſpenſion P may be deduced: for, ſince the angular celerities, produced in the ſyſtem, and in the ſingle body S, by the equal forces $\overline{A+B+C}\times\frac{Pg}{PG}$ and $S\times\frac{Ps}{PS}$, are the ſame, it is manifeſt that the abſolute celerity produced in G, during any given time, will be but the $\frac{PG}{PS}$ part of that produced in S; ſo that only the $\frac{PG}{PS}$ part of the gravity of the ſyſtem is employed in accelerating its motion, the other part $\frac{GS}{PS}$ being loſt on the axis of ſuſpenſion; which axis will therefore, in a direction perpendicular to PG, ſuſtain a force expreſſed by $\overline{A+B+C}\times\frac{Pg}{PG}\times\frac{GS}{PS}$. But this is not the only force by which the axis is affected; ſince, beſides the other part of the force of gravity $(\overline{A+B+C}\times\frac{Gg}{PG})$ in the direction GP, the centrifugal force, acting in the ſame direction, is to be taken into the conſideration; whereof the quantity will be the ſame, as if the whole maſs of the ſyſtem was to be placed in its common center of gravity *G*. For, if upon PS the perpendiculars A*a*, B*b*, C*c* be Fig. 25. let fall; then, the centrifugal forces of the ſeveral bodies A, B, C being as the maſſes drawn into the reſpective diſtances from the center P, the effect of thoſe forces in the direction PG, will therefore be expreſſed by $A \times Pa + B \times Pb + C \times Pc$, which (by the property of the center of gravity) is known to be equal to $\overline{A + B + C} \times PG$.

But the preſſure on the axis may be otherwiſe deduced, independent of the center of oſcillation: for the angular celerity generated in the ſyſtem about its center of gravity G (which is the ſame with the angular celerity about the point of ſuſpenſion P) is intirely the effect of the action on the point of ſuſpenſion; and the momentum, or force, ſufficient to produce that celerity, is found (by the Propoſition) to be $v \times \frac{A \times AG^2 + B \times BG^2 + C \times CG^2}{PG^2}$; which is to the abſolute momentum $v \times \overline{A + B + C}$, generated in the ſyſtem, as $\frac{A \times AG^2 + B \times BG^2 + C \times CG^2}{\overline{A + B + C} \times PG}$ to PG. Therefore the force acting on the axis of ſuſpenſion, in a direction perpendicular to PG, muſt be to the force employed in accelerating the motion of the ſyſtem (in the like direction), in the ſame proportion above ſpecified; ſo that, to have the true meaſure of each, the force of gravity muſt be divided in that ratio: whence (taking $GS = \frac{A \times AG^2 + B \times BG^2 + C \times CG^2}{\overline{A + B + C} \times PG}$) it will be, as $GS + PG$ (PS) is to GS, ſo is the force of gravity, in a direction perpendicular to PG, to the force acting on the axis of ſuſpenſion, in the like direction.

That the proportion here determined is the ſame with that found above, and the point S, the center of oſcillation, is thus made to appear.

$$\begin{aligned} \text{Since } AP^2 &= GP^2 + GA^2 - 2GP \times Ga, \\ BP^2 &= GP^2 + GB^2 + 2GP \times Gb, \\ CP^2 &= GP^2 + GC^2 - 2GP \times Gc, \end{aligned}$$

it is evident that $A \times AP^2 + B \times BP^2 + C \times CP^2$ (as given above) will be $= \overline{A + B + C} \times GP^2 + A \times GA^2 + B \times GB^2 + C \times GC^2 - 2GP \times \overline{A \times Ga - B \times Gb + C \times Gc} = \overline{A + B + C} \times GP^2 + A \times GA^2 + B \times GB^2 + C \times GC^2$, *barely*; becauſe (from the property of the center of gravity) all the quantities $A \times Ga - B \times Gb + C \times Gc$ deſtroy one another. Hence, by ſubſtituting the quantity here found, inſtead of its equal (*in Cor. V.*) we get $SP = \frac{\overline{A+B+C} \times GP^2 + A \times GA^2 + B \times GB^2 + C \times GC^2}{\overline{A+B+C} \times GP}$ $= GP + \frac{A \times GA^2 + B \times GB^2 + C \times GC^2}{\overline{A+B+C} \times GP}$; and conſequently (SG) the diſtance of the center of oſcillation, or percuſſion from the center of gravity $= \frac{A \times GA^2 + B \times GB^2 + C \times GC^2}{\overline{A+B+C} \times GP}$, *the very ſame as above*. Hence it alſo appears, that, if the plane of the motion remains unchanged, the rectangle under SG and GP will be a conſtant quantity; and that, if S be made the point of ſuſpenſion, then P will become the center of oſcillation; and, laſtly, that the oſcillations will be performed in the ſhorteſt time poſſible, when SG and GP are equal to one another, and equal, each, to $\sqrt{\frac{A \times GA^2 + B \times GB^2 + C \times GC^2}{A+B+C}}$; it being well known, that the ſum of two lines, whoſe rectangle is given, will be a *minimum* when the lines themſelves are equal to each other.

The ſame method laid down above, for finding the preſſure upon the axis of ſuſpenſion at reſt, anſwers equally when that axis is ſuppoſed to have a motion, or when the ſyſtem, or body, has a progreſſive motion, as well as an angular one (as is the caſe of a cylinder, which, in its deſcent, is made to revolve about its axis, by means of a rope wrapped about it, whereof one end is made faſt at the place from whence the motion commences): the momentum of the rotation about the center of gravity, generated in a given particle of time, being always as the force producing it, drawn into the diſtance of the point where the force acts, from the center of gravity, as well when that point is in motion, as when it is at reſt.

Another thing it may be proper to take notice of, which is, that in the foregoing conſiderations the bodies A, B, C are ſuppoſed to be very ſmall; ſo as to have all their parts, nearly, at the ſame diſtance from the axis of motion. But, to have the concluſion accurately true, every particle of matter in the ſyſtem ought to be conſidered, and treated, as a diſtinct body: from whence, by means of the *method of fluxions*, the ſum of all the *momenta* will be truly found: but this relating merely to matters of calculation, I have no deſign to touch upon it here. I ſhall only add, that the center of oſcillation may be otherwiſe, very readily, computed, *from Corol. I.* even in caſes where the forces acting on the bodies A, B, C have any given relation to each other. For, if *a*, *b*, *c* be taken to repreſent the, reſpective, meaſures of the ſaid forces (or the *momenta* they would produce in a given time) it is evident, *from thence*, that the angular celerity that would be generated in the ſyſtem (at the diſtance 1, from the center, during the ſame time) will be truly expreſſed by $\frac{a \times AP + b \times BP + c \times CP}{A \times \overline{AP}^2 + B \times \overline{BP}^2 + C \times \overline{CP}^2}$: which, in caſe of a ſingle body S, acted on by the force *s*, becomes $\frac{s \times SP}{S \times \overline{SP}^2}$ (or $\frac{s}{S \times SP}$). Therefore, by putting this laſt value equal to the former, we have $SP = \frac{s}{S} \times \frac{A \times \overline{AP}^2 + B \times \overline{BP}^2 + C \times \overline{CP}^2}{a \times AP + b \times BP + c \times CP}$; ſhewing at what diſtance from the point of ſuſpenſion the ſingle body S muſt be placed, to acquire, by means of the force *s*, the ſame angular celerity as the ſyſtem itſelf acquires, from the action of all the other forces given.

LEMMA.

If a given angle AOB *be divided into two parts* AOC, BOC, *the product (or ſolid) contained under the ſquare of the ſine* (CD) *of the one part* AOC, *and the ſine* (CE) *of the other* BOC, *will be a* maximum, *when the tangent* FC *of the former part is double the tangent* GC *of the latter, or when the ſine of the difference of the parts, is one-third of the ſine of the whole given angle.*

Fig. 26. For, if the ſine (CD) of the former part be denoted by x, and that (CE) of the latter by y, it is well known that ($\dot{x}$) the celerity of x's increaſe (ſuppoſing C to move from A to B) will be in proportion to the celerity ($-\dot{y}$) of y's decreaſe, as the co-ſine

co-ſine of FCD to the co-ſine of GCE; that is, as $\frac{x}{FC}$ to $\frac{y}{GC}$. But, when x^2y is a *maximum*, we have $2x\dot{x}y + x^2\dot{y} = 0$, and conſequently $\dot{x} : -\dot{y} :: x : 2y$. Hence, by equality, $\frac{x}{FC} : \frac{y}{GC} :: x : 2y$; and therefore $FC = 2GC$. Let, now, OH be drawn to biſect FC in H, and let HM and GN be perpendicular to FO; then, ſince it is proved that $FC = 2GC$, it follows that $FH = \frac{1}{3}FG$, and that HM, by ſimilar triangles, muſt likewiſe be $= \frac{1}{3}GN$: but HM and GN are ſines of the angles HOM and AGO, to the equal radii OH and OG; whence the latter part of the *Lemma* is alſo manifeſt.

PROBLEM II.

Suppoſe that a plane ABC, *moving with a velocity and direction repreſented by* bB, *is acted on by a medium, or fluid, whoſe particles move with a velocity repreſented by* DB, *and in directions parallel thereto; to determine the effect of the fluid on the plane, in the direction of its motion* BH, *and alſo what the angle of inclination* ABD *muſt be, that the effect may be the greateſt poſſible.*

Becauſe a particle, impinging on the plane at B, moves thro' the ſpace DB in the time that the plane itſelf, from *abc*, arrives at the poſition ABC, it is evident that the diſtance (D*e*) of the ſaid particle from the plane (produced), at the beginning of that time, will be the meaſure of the relative celerity wherewith the particles of the fluid approach the plane in a direction perpendicular thereto; and, conſequently, that the force of the ſtream in that direction, will be as $\overline{De}^2$ (it being well known that the force of a ſtream upon any plane-ſurface, is always as the ſquare of the relative celerity with which the particles approach it, in a perpendicular direction). Hence, by the reſolution of forces, it will be, as the radius, is to the ſine of the angle ABH (or *ab*H), ſo is the force $\overline{De}^2$, to its required efficacy in the propoſed direction BH. Fig. 27.

Moreover, with regard to the latter part of the Problem, the angle *b*BD, which the directions of the two motions make with each other, being given, as well as the ſides B*b*, BD containing it, the remaining angle B*b*D will from thence be known, as likewiſe D*b*: and ſo D*e* being the ſine of the angle D*be*, to the given radius D*b*, the effect $(\overline{\mathrm{D}e}|^2 \times \text{ſin. } ab\mathrm{H})$ will therefore be a *maximum*, when $\overline{\text{ſin. D}be}|^2 \times \overline{\text{ſin. } ab\mathrm{H}}$ is a *maximum*; that is *(by the Lemma)*, when the ſine of the difference of the angles D*be*, *ab*H, is equal to $\frac{1}{3}$ part of the ſine of the whole given angle B*b*D: from whence the difference being given, the angles themſelves will be known.—The geometrical conſtruction from hence, is extremely eaſy; for, having from the center *b*, with any radius, deſcribed the arch *mr*, on *rb* produced (if neceſſary) let fall the perpendicular *mp*; take $pq = \frac{1}{3}$ of *mp*, and draw *qs* parallel to *pr*, cutting the circle in *s*; then biſect the arch *ms* by the line *bae*, and the thing is done: for the ſine *sv* of S*r* (or of the difference of the angles D*be*, *ab*H) is by conſtruction $(= pq) = \frac{1}{3}$ of *mp* the ſine of the whole given angle B*b*D; as it ought to be, *by the Lemma*.

But, if you had rather have a general Theorem expreſſed in algebraic terms, then let the velocity (*b*B) of the plane be put $= a$, and that (DB) of the fluid $= b$; and let the ſine, and co-ſine of the given angle DB*b* (to the radius 1) be denoted by *m*, and *n*, reſpectively; alſo, having drawn BFL perpendicular to *b*F*e*, put $b\mathrm{F} = x$, and $\mathrm{BF} = y$; then, ſince FB and FL are tangents of the angles F*b*B and F*b*L, to the common radius *b*F, it appears, *by the Lemma*, that FL is $(= 2\mathrm{BF}) = 2y$; whence (ſuppoſing LR and DQ to be perpendicular to B*b*Q, we have (by ſimilar triangles) as B*b* (a) : BF (y) : : BL $(3y)$: $\mathrm{BR} = \frac{3yy}{a}$, and therefore *b*R $(\mathrm{BL} - \mathrm{B}b) = \frac{3yy - aa}{a}$. Alſo, B*b* (a) : *b*F (x) : : BL $(3y)$: $\mathrm{LR} = \frac{3xy}{a}$. But the value of DQ being mb, and that of $b\mathrm{Q} = nb - a$, we have again, by ſim. triang.) $mb : nb - a :: \frac{3xy}{a} : \frac{3yy - aa}{a}$, and conſequently

$$3yy - aa = \frac{nb - a}{mb} \times 3xy, \text{ or } 3yy - yy - xx = \frac{nb - a}{mb} \times 3xy$$

(becauſe $yy + xx = aa$); whence $\overline{\frac{x}{y}}\Big|^{2} + \frac{nb - a}{mb} \times \frac{3x}{y} = 2$, and from thence, by completing the ſquare and extracting the root, $\frac{x}{y} = \sqrt{2 + \frac{9}{4} \times \overline{\frac{nb - a}{mb}}\Big|^{2}} - \frac{3}{2} \times \frac{nb - a}{mb}$, equal to the tangent of the angle *b*BF, the complement of the required angle *ab*H, or ABH. *Q. E. I.*

COROLLARY I.

If the given angle DB*b* be a right-one (which is the caſe when regard is had to the wind ſtriking againſt the ſails of a windmill); then, m being $= 1$, and $n = 0$, our expreſſion for the tangent of *b*BF (which here is equal to the angle of inclination ABD) will become $\sqrt{2 + \frac{9aa}{4bb}} + \frac{3a}{2b}$; and this, if a be taken $= 0$, or the plane be ſuppoſed at reſt, will be $= \sqrt{2}$, barely; anſwering to an angle of 54° 44′. But if the velocity of the plane be ſuppoſed $\frac{1}{8}$, $\frac{1}{4}$, or $\frac{1}{2}$ of the velocity of the medium or ſtream, then the angle of inclination ABD will be found from hence equal to 58° 14′, 61° 27′, or 66° 58′, reſpectively; ſo that, the greater the velocity of the plane is, the greater alſo will be the angle of inclination. Hence it appears that the ſails of a windmill, that the effect may be the greateſt, ought to be more turned towards the wind in the extream parts where the motion is ſwifteſt, than in the parts nearer to the axis of motion; in ſuch ſort, that the tangent of the angle formed by the direction of the wind and the ſail, may be, every-where, equal to $\sqrt{2 + \frac{9aa}{4bb}} + \frac{3a}{2b}$; the velocity a being proportional to the diſtance from the axis of motion.

COROLLARY II.

If, inſtead of the angle DBH (or DB*b*), the angle DBA, which the direction of the ſtream makes with the plane, be given; then it will appear, that the effect will, in this caſe, be a *maximum*, when the ſine of the angle ABH, made by the plane and the direction of its motion, is to the ſine of the ſaid given angle DBA, in the given proportion of $\frac{1}{3}$BD to B*b*. For, the force in the perpendicular direction FB being expreſſed by $\overline{De}^2$, its effect in the direction BH will, therefore, be defined by $\overline{De}^2 \times \frac{BF}{Bb}$, or its equal $\frac{\overline{De}^2 \times Ee}{Bb}$ (ſuppoſing BA produced to meet D*e*E in E). Now DB and the angle DBE (as well as B*b*) being ſuppoſed given, DE is given from thence. But it is well known, that the ſquare of one part of a given line, drawn into the other part, will be a *maximum*, when the former part is the double of the latter. Conſequently D*e* muſt here be the double of E*e*; which laſt, or its equal BF, will therefore be $= \frac{1}{3}$DE.

But, ſin. B*b*F : radius :: BF ($\frac{1}{3}$DE) : B*b*;
And, radius : ſin. DBA :: BD : DE;

by compounding of which, we have the proportion above laid down. But that proportion, it may be obſerved, can only take place when B*b* is equal to, or greater than $\frac{1}{3}$ of DE: for, when B*b* is leſs than $\frac{1}{3}$ of DE, E*e* (which is always leſs than B*b*) cannot be equal to $\frac{1}{3}$ of DE; but will approach the neareſt to it, when BF coincides with B*b*, that is, when the angle F*b*H, or ABH is a right-one; and in this caſe, the effect will be a *maximum*, when the direction of the motion is perpendicular to the plane.—If the given angle DBA be a right-one (which poſition appears from hence to be the moſt advantageous, becauſe DE then becomes = DB) it follows that the ſine of the angle ABH, which the required direction makes with the plane, will be to the radius, as $\frac{1}{3}$ part of the velocity of the ſtream is to the velocity of the plane (or ſail).—Hence, if the force of the wind be capable of producing a degree of celerity in a ſhip, greater than $\frac{1}{3}$ part of its own celerity, it is evident that the ſhip may run ſwifter upon an oblique courſe, than when ſhe ſails directly before the wind*.

PROBLEM III.

Suppoſe that a thread ACnCA, *having two equal weights* A, A, *ſuſpended at the ends thereof, is hung over two tacks* C, C, *in the ſame horizontal line; and that to the middle point of the thread* (n) *equally diſtant from the tacks, another given weight* B *is fixed, which is permitted to deſcend by its own gravity, ſo as to cauſe the other two weights, at the ſame time, to aſcend: it is propoſed to find the law of the velocity by which the ſaid weights aſcend and deſcend; abſtracting from the reſiſtance of the air, the weight of the thread, and the friction on the tacks.*

Let v denote the velocity of B (meaſured by the diſtance Fig. 28. that might be uniformly gone over in one ſecond of time), and let b ($= 32\frac{1}{6}$ feet) $=$ the meaſure of the velocity which gravity can generate in a falling body, in one *ſecond*; putting CE $= a$, E$n = x$, C$n = y$, and the tenſion of the thread $= w$: then $\frac{\dot{x}}{v}$ being the time in which B would, uniformly, deſcribe the diſtance $\dot{x}$, we ſhall have, as 1 (ſecond) is to $\frac{\dot{x}}{v}$, ſo is b (the velocity generated by gravity in one ſecond) to, $\frac{b\dot{x}}{v}$, the velocity generated (or deſtroyed) by gravity in the time $\frac{\dot{x}}{v}$.

Moreover it will be, as BC (y) : En ($\sqrt{yy-aa}$) :: v : $\frac{v\sqrt{yy-aa}}{y} =$ the velocity with which the weights A, A aſcend; whoſe fluxion $\frac{\overline{yy-aa}\times y\dot{v}+aav\dot{y}}{yy\sqrt{yy-aa}}$ $\left(= \frac{\overline{yy-aa}\times y\dot{v}+aav\dot{y}}{yyx}\right)$ is therefore the increaſe of that velocity, in the time $\frac{\dot{x}}{v}$: but were not the ſtring to act on the ſaid weights, their velocity (inſtead of being increaſed) would be diminiſhed, and that by the quantity $\frac{b\dot{x}}{v}$ (as is found above). Therefore the whole alteration of motion ariſing from the tenſion of the ſtring is to that ariſing from the action of gravity, in the proportion of $\frac{b\dot{x}}{v} + \frac{\overline{yy - aa} \times y\dot{v} + aav\dot{y}}{yyx}$ to $\frac{b\dot{x}}{v}$; and, conſequently, the tenſion of the ſtring (w) will be to the weight of the body A, in the ſame proportion: whence we have $w = \mathrm{A} \times \overline{1 + \frac{\overline{yy - aa} \times yv\dot{v} + aavv\dot{y}}{byyx\dot{x}}}$.

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* In the above conſiderations the velocity of the plane (or ſail) is, all along, treated as a given quantity; becauſe the ſame direction that gives the effective force the greateſt, when the velocity is given, muſt neceſſarily give the velocity the greateſt poſſible, when the force, alone, is given.

Again, it will be (by the reſolution of forces) as Cn (y) is to En (x), ſo is $2w$ (the double of the tenſion of the thread) to $\frac{2wx}{y}$, the effect of that tenſion to retard the deſcent of the weight B; which being ſubtracted from the gravity (B), the remainder $\mathrm{B} - \frac{2wx}{y}$ will be the force by which B's motion is accelerated. Hence we have, as B is to $\mathrm{B} - \frac{2wx}{y}$, ſo is $\left(\frac{b\dot{x}}{v}\right)$ the velocity that would be generated by the gravity in the time $\frac{\dot{x}}{v}$, to that ($\dot{v}$) generated in the ſame time, by the force $\mathrm{B} - \frac{2wx}{y}$. Prom whence, by multiplying extreams and means, we get $v\dot{v} = b\dot{x} - \frac{2bx\dot{x}w}{\mathrm{B}y} = b\dot{x} - \frac{2b\mathrm{A}x\dot{x}}{\mathrm{B}y} - 2\mathrm{A} \times \frac{\overline{yy - aa} \times yv\dot{v} + a^2v^2\dot{y}}{\mathrm{B}y^3}$ (by ſubſtituting the value of w) $= b\dot{x} - \frac{2b\mathrm{A}\dot{y}}{\mathrm{B}} - \frac{2\mathrm{A}v\dot{v}}{\mathrm{B}} + \frac{2a^2\mathrm{A}}{\mathrm{B}} \times \frac{yv\dot{v} - v^2\dot{y}}{y^3}$ (becauſe $y\dot{y} = x\dot{x}$) $= b\dot{x} - \frac{2b\mathrm{A}\dot{y}}{\mathrm{B}} - \frac{2\mathrm{A}v\dot{v}}{\mathrm{B}} + \frac{a^2\mathrm{A}}{\mathrm{B}} \times \frac{2y^2v\dot{v} - 2v^2y\dot{y}}{y^4}$: and, conſequently, by taking the fluent, $\frac{v^2}{2} = bx - \frac{2b\mathrm{A}y}{\mathrm{B}} - \frac{\mathrm{A}v^2}{\mathrm{B}} + \frac{a^2\mathrm{A}}{\mathrm{B}} \times \frac{vv}{yy} + d$ (d being the neceſſary correction); which equation, if $\frac{\mathrm{B}}{2\mathrm{A}}$ be put $= m$, will be reduced to $v = y\sqrt{\frac{2bmx - 2by + 2dm}{\overline{m + 1} \,.\, yy - aa}}$; ſhewing the true velocity of the body B; whence that of the body A ($= x\sqrt{\frac{2bmx - 2by + 2dm}{\overline{m+1}\,.\,yy - aa}}$) will alſo be known. *Q. E. I.*

COROLLARY I.

If the firſt values of x and y, when the motion commences, be expreſſed by f and g, reſpectively; then, v being $= 0$, when $x = f$, and $y = g$, we ſhall have $0 = 2bmf - 2bg + 2dm$, and conſequently $2dm = -2bmf + 2bg$; ſo that the general value of v will be $y\sqrt{\frac{2bmx - 2by - 2bmf + 2bg}{\overline{m+1}\,.\,yy - aa}} = y\sqrt{\frac{2bm \times \overline{x-f} - 2b \times \overline{y-g}}{\overline{m+1}\,.\,yy - aa}}$. From whence the greateſt diſtance through which the weight B can deſcend, before its whole motion is deſtroyed by the other weights A, A, may be eaſily determined: for, ſince the velocity, at the loweſt point of the deſcent, vaniſhes, or becomes equal to nothing, we ſhall, in that circumſtance, have $2bm \times \overline{x-f} - 2b \times \overline{y-g} = 0$, or $m \times \overline{x-f} + g$ ($= y = \sqrt{\overline{xx+aa}}$) $= \sqrt{\overline{xx+gg-ff}}$; which, ſquared, gives $m^2 \times \overline{x-f}|^2 + 2mg \times \overline{x-f} = xx - ff$: whence, dividing by $x - f$, we have $m^2 \times \overline{x-f} + 2mg = x + f$; and conſequently $x = \frac{2mg - \overline{1+mm}\,.\,f}{1 - mm}$; exhibiting the diſtance of the point n below the horizontal line CC, when the whole motion is deſtroyed, and all the weights begin to move the contrary way. But it muſt be obſerved, that this can only happen when m is leſs than unity, or when the weight B is leſs than the ſum of the other two: for, if m be equal to unity, x will be infinite; and, if m be greater than unity, the value of x will come out negative; which ſhews the thing to be impoſſible, or that the weight B muſt continually deſcend; except when m is leſs than unity, or B leſs than 2A: in which laſt caſe, it appears that the bodies will oſcillate, backwards and forwards, continually; in ſuch ſort, that the two extream diſtances from the horizontal line CC will be expreſſed by f and $\frac{2mg - \overline{1+mm}\,.\,f}{1 - mm}$; whereof the latter, when $f = 0$, will become $\frac{2ma}{1 - mm} =$

$\frac{m}{1 - mm} \times CC$; ſhewing the loweſt deſcent of n, from the line CC, when the motion commences from that line.—By making f and $\frac{2mg - \overline{1 + mm} \, . \, f}{1 - mm}$ equal to each other, we get $f = mg = \frac{B}{2A} \times g$, or $f : g :: B : 2A$. From which it appears, that, if the firſt poſition of the weight B be ſuch, that En is to Cn in the given proportion of B to 2A, no motion at all will enſue, but the weights remain in *equilibrio*. Whence it is evident, that, if the motion commences from any point below *that* here determined, the weight B will firſt of all aſcend, till the diſtance from CC is $\frac{2mg - \overline{1 + mm} \, . \, f}{1 - mm}$; after which it will again deſcend, to its firſt diſtance f; and ſo on, backwards and forwards, continually.

COROLLARY II.

If CnC, in the firſt poſition of B, be ſuppoſed to coincide with the horizontal-line CEC, and the body B be impelled from thence with any given celerity c (meaſured, as above, by the ſpace that would be uniformly gone over in one ſecond of time); then, v being $= c$, when $x = 0$ and $y = a$, we ſhall, by ſubſtituting theſe values in the general equation $\left(v = y\sqrt{\frac{2bmx - 2by + 2dm}{\overline{m + 1} \, . \, yy - aa}}\right)$ obtain $c = a\sqrt{\frac{-2ba + 2dm}{maa}} = \sqrt{\frac{2dm - 2ba}{m}}$, and conſequently $2dm = mc^2 + 2ba$; ſo that v is *here* $= y\sqrt{\frac{2bmx - 2by + 2ba + mc^2}{\overline{m + 1} \, . \, yy - aa}}$; which, when $b = 0$, or, when the bodies are not acted on by gravity, will become $v = \frac{m^{\frac{1}{2}}cy}{\sqrt{\overline{m + 1} \, . \, yy - aa}}$. And in this caſe, the time of moving thro' En (whereof the fluxion is $\frac{\dot{x}}{v} = \frac{\dot{x}\sqrt{\overline{m + 1} \, . \, yy - aa}}{m^{\frac{1}{2}}cy} = \frac{\dot{x}\sqrt{\overline{m + 1} \, . \, xx + ma^2}}{m^{\frac{1}{2}}c\sqrt{xx + aa}}$) may be readily found, by means of an *hyperbola*, whoſe tranſverſe and conjugate axes are $\frac{2a}{\sqrt{m}}$ and $2a$; *it* being in proportion to the time of moving uniformly over the ſame diſtance (x) with the given celerity at E, as the arch of the *hyperbola* is to its ordinate x.

PROBLEM IV.

Suppoſing a ſpherical body of ice, or any other matter, revolving about its axis, to be reduced to a ſtate of fluidity; to determine the change of figure thence ariſing.

It is demonſtrable, that the figure of an homogeneous fluid, revolving about an axis (PS), having all its particles quieſcent with regard to each other, muſt be *that* of an oblate ſpheroid OAPES *(ſee Art.* 395 *of my Doctrine of Fluxions)*; and that the particular ſpecies of ſuch ſpheroid, anſwering to any given time of revolution p, will be truly defined by the equation $p = q\sqrt{\frac{\frac{2}{3}t^3}{\overline{3+tt}\times A - 3t}}$; where $1 : 1 + tt :: PS^2 : AE^2$; PS being the axis, and AE the equatoreal diameter; alſo $A =$ the circular arch, whoſe radius is unity, and tangent t; and $q =$ the time wherein a ſolid ſphere, of the ſame magnitude and denſity with the ſpheroid, muſt revolve, ſo that the centrifugal force at the equator thereof, may be exactly equal to the attraction, or gravity. Now it is evident, that, whatſoever figure a fluid, revolving about an axis, at any time hath, the momentum of rotation about the axis will be no-ways changed, with the figure, by the action of the particles on each other: ſo that the momentum of our propoſed fluid, ariſing from the ſphere of ice, will, at all times, be the very ſame with that of the ſphere itſelf. From whence it may be eaſily proved, that the time wherein one intire revolution of the fluid, conſidered as a ſpheroid, might be uniformly performed, muſt be always as AE^2: therefore, if we make $e = AE$, and put $d =$ the diameter of the ſphere (or of the fluid, when $AE = PS$) it follows that the ſaid time * will be truly expreſſed by

Fig. 29.

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* At Art. 399. of my Fluxions, this time is, by miſtake, put down $= \frac{e}{d} \times s$ (inſtead of $\frac{e^2}{d^2} \times s$); whereby the remaining part of that Article is rendered erroneous.

by $\frac{ee}{dd} \times s$ (ſuppoſing s to denote the given time of revolution of the body, when under the form of a ſphere): which being put $\left(= q\sqrt{\frac{\frac{2}{3}t^3}{\overline{3+tt} \times \mathrm{A} - 3t}}\right)$ the time wherein the revolution is performed, when the particles are in *equilibrio*, we ſhall thence have $\frac{\overline{3+tt} \times \mathrm{A} - 3t}{t^3} = \frac{2q^2}{3s^2} \times \frac{d^4}{e^4}$. But, becauſe PS $\left(= \frac{\mathrm{AE}}{\sqrt{1+tt}}\right) = \frac{e}{\sqrt{1+tt}}$, the maſs of the ſpheroid will therefore be as $\frac{e^3}{\sqrt{1+tt}}$ $(= \mathrm{AE}^2 \times \mathrm{PS})$; and *that* of the ſphere, as d^3: which two quantities being made equal to each other, we have $\frac{d^4}{e^4} = \frac{1}{\overline{1+tt}|^{\frac{2}{3}}}$. And, this value being wrote in the room of its equal, we have $\frac{\overline{3+tt} \times \mathrm{A} - 3t}{t^3} = \frac{2q^3}{3s^2} \times \frac{1}{\overline{1+tt}|^{\frac{2}{3}}}$, or $\frac{\overline{3+tt} \times \mathrm{A} - 3t}{t^3} \times \overline{1+tt}|^{\frac{2}{3}} = \frac{2q^2}{3s^2}$. From the reſolution of which equation the value of t, and the ſpheroid itſelf, will be known. The ſpheroid thus determined, is *that* under which the fluid might remain in *equilibrio*, were the particles to be, once, quieſcent with reſpect to each other: but the particles, in their receſs from the axis, do, through the centrifugal force, acquire a motion from the axis, which is not immediately deſtroyed, on the fluid's aſſuming the figure, or degree of oblateneſs above determined; the equatoreal parts ſtill continuing to recede from the axis, till the gravitation, by degrees, prevails, and in the end quite overcomes the ſaid motion. After which the equatoreal parts will begin to ſubſide, and again approach the axis, in the very ſame manner they before receded therefrom: and ſo will continue oſcillating, backwards and forwards, *ad infinitum*. But if the fluid is ſuppoſed to have ſome degree of tenacity, the oſcillations will be, every time, contracted, and the parts of the fluid will then converge to an equilibrium, under the form above determined.

LEMMA.

Suppoſing a body to move with an uniform celerity, in a right-line AD; *to determine the rate of increaſe of the relative celerity by which* it *recedes from a given point* C, *out of that line.*

Make CA (perpendicular to AD) $= a$, and AB $= x$; and let the meaſure of the body's celerity, or the ſpace gone over in a given time g, be denoted by c: then will $\frac{g\dot{x}}{c}$ expreſs the time of deſcribing $\dot{x}$ (or Bb); and it is well known, that $c \times \frac{x}{\sqrt{xx+aa}}$ $(= c \times \frac{\text{AB}}{\text{BC}})$ will be the true meaſure of the celerity with which CB increaſes; whoſe fluxion, $\frac{ca^2\dot{x}}{\overline{xx+aa}|^{\frac{3}{2}}}$, is therefore the (uniform) increaſe of that celerity, in the time $\frac{g\dot{x}}{c}$: hence it will be, as $\frac{g\dot{x}}{c} : g$ (the time given) $:: \frac{ca^2\dot{x}}{\overline{xx+aa}|^{\frac{3}{2}}} :$ $\frac{c^2a^2}{\overline{xx+aa}|^{\frac{3}{2}}}$ $\left(= \frac{c^2 \times \overline{\text{AC}}|^2}{\overline{\text{CB}}|^3}\right)$ the required increaſe, that would uniformly ariſe in the given time g: which increaſe, ſince $\frac{c \times \text{AC}}{\text{CB}}$ repreſents the paracentric velocity of the body (in a direction perpendicular to CB) will be, always, expreſſed by the ſquare of the meaſure of the body's paracentric velocity, applied to the diſtance (BC) from the given point, or center. *Q. E. I.* Fig 30.

COROLLARY.

It is evident from hence, that if a force, which in the given time g is ſufficient to generate the ſaid increaſe of velocity, be ſuppoſed to urge the body towards the center C, and thereby deflect it from its rectilineal motion, the celerity with which CB increaſes will *then* be uniform; becauſe the force applied, each moment of time, is juſt ſufficient to deſtroy the increaſe that would ariſe, in the ſame moment, from the body's being ſuffered to continue its motion uniformly in a right-line.—If the direction (Bb) of the motion is perpendicular to CB, the body, thus acted on (as no celerity is generated in the direction CB), will move in the circumference of a circle. Conſequently

the

the force above determined is the ſame with the centrifugal force in a circle, when the diſtance from the center, and the angular celerity are the ſame.

But all this may be made to appear in a different manner, by ſuppoſing B*b* exceeding ſmall: for, if *b*E be made perpendicular to CB (produced), BE will then expreſs the length whereby CB would be uniformly augmented, in the time $\left(\frac{g\times Bb}{c}\right)$ of deſcribing B*b*; and therefore *eb*, the exceſs of C*b* above CE, will be the ſpace through which the force muſt cauſe the body to deſcend, in order that the increaſe of the diſtance from the center C may be the ſame as would uniformly ariſe with the firſt celerity, at B. But it is evident that this exceſs *eb* (which, by the property of the circle, is $= \frac{\overline{Eb}^2}{2CE}$, or $\frac{\overline{Eb}^2}{2CB}$) alſo expreſſes the effect of the force, neceſſary to cauſe a body to revolve in the circumference of a circle E*ef*, with the ſame angular celerity.—To determine, from hence, the velocity which this force would generate in the given time *g*, we have, as $\frac{g^2\times\overline{Bb}^2}{cc}$ (the ſquare of the time of deſcribing B*b*, or B*e*) is to g^2, ſo is $\frac{\overline{Eb}^2}{2CB}$ to $\left(\frac{c^2\times\overline{Eb}^2}{2CB\times\overline{Bb}^2}\right)$ the ſpace through which the ball might fall, by means of the ſaid force, in the given time *g*; the double of which, $\frac{c^2\times\overline{Eb}^2}{CB\times\overline{Bb}^2}$ (or its equal $\frac{c^2\times\overline{AC}^2}{CB\times\overline{CB}^2}$) is, therefore, the true meaſure of the velocity ſought; becauſe the diſtance gone over by a falling body is but the half of that which might be deſcribed in the ſame time, with the velocity acquired at the end of the deſcent.—The quantity here determined (as has been before obſerved) is the meaſure of the force by which the body is made to recede from C with an uniform celerity: if a force, leſs or greater than this, be ſuppoſed to act, the difference will cauſe an increaſe or decreaſe of celerity in the line CB, proportional to the ſaid difference.

PROBLEM V.

Suppoſe that a body, let go from a given place A, *in a given direction, with a given celerity, is continually ſolicited towards a given point* C, *by a given centripetal force*; *to determine the path* ABP *in which the body will move.*

From the center C, through A, let the circumference of a circle ADK be deſcribed; and, ſuppoſing B to repreſent the place of the body, Fig. 31.

the radius CD of the circle . . . . = a,
the *radius vector* CB . . . . = x,
the arch AD, meaſuring the angle ACB . . . . = z,
put the time of deſcribing the angle ACB . . . . . . = t,
the meaſ. of the celerity with which the line CB incr. = v,
the meaſ. of the celer. with which the area ACB incr. = au,
the meaſure of the centripetal force . . . . = Q;

where, by the meaſure of a celerity, I mean the ſpace that would be uniformly deſcribed with that celerity in a given time g; and by the meaſure of a force, I underſtand the meaſure of the celerity that might be uniformly generated by the force, in the ſame given time.

Since the celerity with which the area ACB increaſes is expreſſed by au, it is evident that the paracentric velocity of the *radius vector* CB, at the middle point (b), will be expreſſed by $\frac{au}{x}$, and that of the body itſelf by $\frac{2au}{x}$; the ſquare of which laſt, divided by x, will give $\frac{4a^2u^2}{x^3}$ for the true meaſure of the centrifugal force *(by the Lemma)*; whence it appears that $\left(\frac{4a^2u^2}{x^3} - Q\right)$ the exceſs thereof above the centripetal force Q, is that force whereby the celerity v is accelerated: therefore we have $g : \dot{t} :: \frac{4a^2u^2}{x^3} - Q$ (the meaſure of the celerity generated in the given time g) : $\dot{v}$. But, becauſe the paracentric velocity of the body is $\frac{2au}{x}$, *that* of the point D (deſcribing the circular arch AD) will be $\frac{2au}{x} \times \frac{a}{x}$, or $\frac{2a^2u}{xx}$; and ſo we have $g : \dot{t} :: \frac{2a^2u}{xx}$ (the diſtance deſcribed in the time g) : $\dot{z}$ (the diſtance deſcribed in the time $\dot{t}$): whence, by equality, $\frac{4a^2u^2}{x^3} - Q : \dot{v} :: \frac{2a^2u}{xx} : \dot{z}$; and conſequently $\dot{v} = \overline{\frac{4a^2u^2}{x^3} - Q} \times \frac{x^2\dot{z}}{2a^2u}$.

Again, the ſpaces $\dot{z}$ and $\dot{x}$ (deſcribed in the ſame time) being in the ſame proportion as the celerities $\frac{2a^2u}{xx}$ and v, with which they are deſcribed, we alſo have $v = \frac{2aau\dot{x}}{xx\dot{z}}$. To exterminate v and $\dot{v}$ out of *this*, and the preceding equation, make $\frac{a\dot{x}}{xx} = \dot{w}$, (or $w = -\frac{a}{x} + 1$); then $v = \frac{2au\dot{w}}{\dot{z}}$, and $\frac{2a\dot{u}\dot{w} + 2au\ddot{w}}{\dot{z}} = \dot{v} = \overline{\frac{4a^2u^2}{x^3} - Q} \times \frac{x^2\dot{z}}{2a^2u} = \frac{2u\dot{z}}{x} - \frac{Qx^2\dot{z}}{2a^2u} = \frac{2u\dot{z} \times \overline{1-w}}{a} - \frac{Q\dot{z}}{2u \times \overline{1-w}|^2}$ (by writing $\frac{a}{1-w}$ for its equal x); which equation may be reduced to $\frac{\ddot{w}}{\dot{z}\dot{z}} + \frac{w}{aa} = \frac{1}{aa} - \frac{\dot{u}\dot{w}}{u\dot{z}\dot{z}} - \frac{Q}{4au^2 \times \overline{1-w}|^2}$; expreſſing the general relation of w and z, or of x and z, according to any value of u. But in the caſe propounded, wherein no force is ſuppoſed to act, beſides that tending to the center C, the celerity au with which the area ACB increaſes, will be a conſtant quantity; and therefore, $\dot{u}$ being here $= 0$, our equation becomes $\frac{a^2\ddot{w}}{\dot{z}\dot{z}} = 1 - w - \frac{aQ}{4u^2 \times \overline{1-w}|^2}$: from whence, when Q is given in terms of x (or w), the relation of w and z may be determined.

COROLLARY I.

Hence, if the centripetal force, by which a body deſcribes a given orbit at reſt, be known, the increaſe of that force, when the orbit itſelf is ſuppoſed to have a motion round the center of force, may be eaſily deduced: for, let the angular motion of the orbit, be to that of the body in the orbit, in the conſtant ratio of m to 1; then, the whole angular celerity of the body, here, being in proportion to the angular celerity when the orbit is quieſcent, as $m+1$ to 1, the centrifugal force here, will therefore be to that $\left(\frac{4a^2u^2}{x^3}\right)$ in the quieſcent orbit, in the duplicate ratio of $m+1$ to 1 *(by the Lemma)*, and ſo will be truly expreſſed by $\overline{m+1}|^2 \times \frac{4a^2u^2}{x^3}$. From whence it appears, that $\overline{mm+2m} \times \frac{4a^2u^2}{x^3}$ is the increaſe of the centrifugal force ariſing from the motion of the orbit: which quantity, therefore, muſt be *that* whereby the centripetal force ought to be likewiſe increaſed, in the moveable orbit; ſo that the difference of the two forces, whereby the motion of the body in the line CB is accelerated, may be the ſame here, as in the quieſcent orbit; in which caſe the value of CB itſelf, in all contemporay poſitions, muſt neceſſarily be the ſame. Hence it appears, that the increaſe of the centripetal force, in order to the deſcription of a moveable orbit, will be always inverſely as the cube of the diſtance; and will, moreover, be to the centrifugal force in the quieſcent orbit (in all contemporary poſitions), in the conſtant ratio of $mm+2m$ to 1.

COROLLARY II.

If the centripetal force (Q) be ſuppoſed, inverſely, as the ſquare of the diſtance, and the given value thereof, at the lower apſe A, be to the centrifugal force there, in any given ratio of $1-e$ to 1; then, as the general value $\left(\frac{4a^2u^2}{x^3}\right)$ of the centrifugal force, will, at A, become $=\frac{4u^2}{a}$, the centripetal force there will be expreſſed by $\frac{4u^2}{a} \times \overline{1-e}$; and conſequently that at B by $\frac{\overline{1-e} \times 4u^2}{a} \times \frac{a^2}{x^2}$, or its equal $\frac{\overline{1-e} \times 4u^2}{a} \times \overline{1-w}|^2$. Which value being ſubſtituted for Q, our equation here becomes $\frac{a^2\ddot{w}}{\dot{z}\dot{z}} = e-w$: whence, multiplying by $\dot{w}$ and taking the fluent, we get $\frac{a^2\dot{w}^2}{2\dot{z}\dot{z}} = ew - \frac{w^2}{2}$ (where, the angle CAB being ſuppoſed

Fig. 32. ſuppoſed a right-one, no correction is neceſſary); ſo that we have $\dot{z} = \frac{a\dot{w}}{\sqrt{2ew - ww}}$, or $\frac{e\dot{z}}{a} = \frac{e\dot{w}}{\sqrt{2ew - ww}}$: but the laſt of theſe quantities is known to expreſs the fluxion of a circular arch (A), whoſe verſed-ſine is w and radius e; therefore $\frac{e\dot{z}}{a}$ being $= \dot{A}$, or $a : e :: z : A$, it follows that the arches z and A (which are in the ſame proportion with their radii a, e) muſt be ſimilar, and conſequently their verſed-ſines, AF and w, in the ſame proportion above ſpecified, or as a to e: whence we have $w = \frac{e}{a} \times \text{AF}$, that is, $1 - \frac{\text{AC}}{\text{BC}} = e \times \frac{\text{AF}}{\text{AC}}$. But $\frac{\text{AF}}{\text{AC}} = \frac{\text{AC} - \text{CF}}{\text{AC}} = 1 - \frac{\text{CF}}{\text{CD}} = 1 - \frac{\text{CE}}{\text{CB}}$ (ſuppoſing BE perpendicular to AC); therefore $1 - \frac{\text{AC}}{\text{BC}} = e \times \overline{1 - \frac{\text{CE}}{\text{CB}}}$, and conſequently BC — AC (= BD) $= e \times \overline{\text{CB} - \text{CE}}$; from which equal quantities take away $e \times \text{BD}$, ſo ſhall $\text{BD} - e \times \text{BD} = e \times \text{AE}$, and therefore $1 - e : e :: \text{AE} : \text{BD}$; which is a known property of the conic ſections, with reſpect to lines drawn from the focii. Hence it appears, that the trajectory will be an *ellipſe*, *parabola*, or *hyperbola*, according as the *antecedent* $1 - e$ is greater, equal to, or leſs than the *conſequent* e; or according, as the centripetal force at A, is greater, equal to, or leſs than half the force ſufficient to retain the body in the circular orbit ADK.—As to the particular ſpecies of the curve, correſponding to any given value of e, it is, from hence, very eaſily determined: for, if AO be made to repreſent the ſemi-tranſverſe axis, then will AO : OC (:: AE : BD, *p. conics)* :: $1 - e : e$; therefore, by diviſion, AO : AC :: $1 - e : 1 - 2e$; whence AO is known.

COROLLARY III.

If to the foregoing force, varying in the inverſe ratio of the ſquare of the diſtance, another force, which is inverſely as the cube of the diſtance, be joined (which, at A, is to the former part in any given ratio of s to $1 - e$*)*, the place of a body, thus acted on, may be found in the ſame *conic ſection* Fig. 33. APRS, above determined, ſuppoſing it to have a motion about the focus C, which is to that of the body in the ſection (referred to the ſame point C) in the conſtant ratio of m to 1; the value of m being $= \sqrt{1+s} - 1$, or ſuch, that $mm + 2m : 1 :: s : 1$ (*by Corol. I.*)—If the centripetal force be barely as the cube of the diſtance inverſely, the curve A′B will degenerate to a right-line; in which the body will continue to move with an uniform velocity, while the line itſelf BA′ (always touching the circle in A′) is ſo carried along by the motion of the radius CA′, that the angle ACA′ ſhall be to the angle A′CB, in the conſtant proportion above ſpecified; the ratio of the centripetal, and centrifugal forces at A (and conſequently in every other poſition) being expreſſed by that of s to $1 + s$.

COROLLARY IV.

If the centripetal force to be as any power of the diſtance, whoſe exponent is n, and the given value thereof, at A, be in proportion to the centrifugal force $\left(\frac{4u^2}{a}\right)$, as r to 1; we ſhall then have $Q = \frac{4ru^2}{a} \times \frac{x^n}{a^n} = \frac{4ru^2}{a} \times \frac{1}{\overline{1-w}|^{n}}$; and here our equation, $\frac{a^2\ddot{w}}{\dot{z}\dot{z}} = 1 - w - \frac{aQ}{4u^2 \times \overline{1-w}|^{2}}$, will become $\frac{a^2\ddot{w}}{\dot{z}\dot{z}} = 1 - w - \frac{r}{\overline{1-w}|^{n+2}}$: which being multiplied by $\dot{w}$, and the fluent taken, we thence get $\frac{a^2\dot{w}^2}{2\dot{z}\dot{z}} = w - \frac{w^2}{2} - \frac{r \cdot \overline{1-w}|^{-n-1}}{n+1} + \frac{r}{n+1}$, and conſequently

$$\dot{z} = \frac{a\dot{w}}{\sqrt{2w - w^2 - \frac{2r \cdot \overline{1-w}|^{-n-1}}{n+1} + \frac{2r}{n+1}}};$$

from whence the value of z, by infinite ſeries, or the quadrature of curves, may be found.

But when r differs but little from unity, and the orbit is nearly circular, $\frac{r}{\overline{1-w}|^{n+2}}$ (becauſe of the ſmallneſs of w)

will be nearly equal to $r \times \overline{1 + \overline{n+2} . w}$; and therefore $\frac{a^2 \ddot{w}}{\dot{z}\dot{z}} = 1 - w - r - \overline{n + 2} . rw = 1 - r - \overline{n + 3} . w$, nearly; and conſequently $\frac{a^2 \ddot{w}}{\overline{n+3} . \dot{z}\dot{z}} = \frac{1 - r}{n+3} - w$.

Fig. 33. Put $f = \frac{1-r}{n+3}$, and $A = \sqrt{n+3} \times z$, that is, let A repreſent an arch A′D of the circle ADK, which is always to the arch AD (or z), in the conſtant ratio of $\sqrt{n+3}$ to 1: then $\dot{A}\dot{A}$ being $= \overline{n+3} \times \dot{z}\dot{z}$, our equation, by ſubſtituting theſe values, will become $\frac{a^2 \ddot{w}}{\dot{A}\dot{A}} = f - w$; which differs in nothing from that $\left(\frac{a^2 \ddot{w}}{\dot{z}\dot{z}} = e - w\right)$ reſolved in *Corol. II*, excepting only, that A and f are here uſed, inſtead of z and e: whence it is manifeſt, that the value of w (there repreſented by $e \times$ verſed-ſine of z) will here be truly expreſſed by $f \times$ verſed-ſine of A: from whence and what is *there* demonſtrated, it alſo appears, that the place of the body will be in the periphery of a given ellipſe A′BR, revolving about its focus C, with an angular celerity, which is to that of the body in the ellipſe, in the conſtant ratio of the arch AA′ to the arch A′D, or as $\overline{1 - \sqrt{n+3}}$ to $\sqrt{n+3}$. And it is evident, that the motion of the apſides will be to the motion of the body in the ellipſe, referred to the focus C, in the ſame given ratio; and that the angle deſcribed by the body in moving from one apſide to the other (becauſe AD is always $=$ A′D $\times \frac{1}{\sqrt{n+3}}$) will be $= 180^\circ \times \frac{1}{\sqrt{n+3}} = \frac{180^\circ}{\sqrt{n+3}}$. All which concluſions, as well as thoſe derived in the preceding *Corollaries*, exactly agree with what Sir Isaac Newton has demonſtrated, by a very different method, in the third and ninth *Sections* of the firſt book of his *Principia*.—As to the motion of the *apſides* of the lunar orbit, with the other inequalities depending on the ſun's action, theſe require the uſe of other principles, and the ſolution of the following

PRO-

PROBLEM VI.

The ſame being ſuppoſed as in the laſt Problem, and that, beſides the force tending to the center C, *another force, whoſe meaſure is* R, *acts continually on the body, in a direction perpendicular to the radius-vector* BC; *it is propoſed to determine the curve* ABP *which the body, ſo acted on, will deſcribe.*

Every thing in the preceding Problem being retained, Fig. 31. we have nothing more to do here, than to get an equation for u, by means of the new force, whereon the increaſe or decreaſe of u intirely depends. In order to this, we have, as g (the given time) is to $\dot{t}$, ſo is R, the velocity generated in the time g, to $\frac{R\dot{t}}{g}$, the velocity generated in the time $\dot{t}$, in a direction perpendicular to BC: whence the correſponding increaſe of the celerity au, with which the area ACB is generated, will be expreſſed by $\frac{R\dot{t}}{g} \times \frac{x}{2}$, that is, $\frac{R\dot{t}}{g} \times \frac{x}{2}$ will be $= a\dot{u}$. But, as it has been proved that $g : \dot{t} :: \frac{2a^2u}{xx} : \dot{z}$, we have $\frac{\dot{t}}{g} = \frac{x^2\dot{z}}{2aau}$; and therefore $a\dot{u} = \frac{Rx^3\dot{z}}{4aau}$, or $2u\dot{u} = \frac{\frac{1}{2}Rx^3\dot{z}}{a^3} = \frac{\frac{1}{2}R\dot{z}}{\overline{1-w}|^{3}}$ (becauſe $x = \frac{a}{1-w}$). Hence, by taking the fluent, we have $u^2 = c^2 + \text{flu.}\frac{\frac{1}{2}R\dot{z}}{\overline{1-w}|^{3}}$ (c^2 being put for the neceſſary correction, or the value of u^2 when $z = 0$). From which equation, and that $\left(\frac{\ddot{w}}{\dot{z}\dot{z}} + \frac{w}{aa} = \frac{1}{aa} - \frac{\dot{u}\dot{w}}{u\dot{z}\dot{z}} - \frac{Q}{4au^2 \times \overline{1-w}|^{2}}\right)$ derived by the preceding Problem, the relation of u, w, and z may be determined, when the law of the forces Q and R is aſſigned. *Q. E. I.*

COROLLARY

If the forces *Q* and *R* are ſuppoſed to be in proportion to $\left(\frac{4cc}{a}\right)$ the centrifugal force at A, as Δ to 1, and Π to 1, reſpectively (Δ and Π being any variable quantities whatever), and if the celerity (au) with which the area ACB increaſes be ſuppoſed in proportion to (ac) the firſt value thereof at A, as $\Sigma^{\frac{1}{2}}$ to 1; then, Q being $= \frac{4cc\Delta}{a}$, $R = \frac{4cc\Pi}{a}$, and $u = c\,\Sigma^{\frac{1}{2}}$, our two equations, by ſubſtituting theſe values, will become $\Sigma = 1 + \text{flu.}\frac{2\Pi\dot{z}}{a\times\overline{1-w}^{3}}$, and $\frac{\ddot{w}}{\dot{z}\dot{z}} + \frac{w}{aa} = \frac{1}{aa} - \frac{\dot{\Sigma}}{2\Sigma}\times\frac{\dot{w}}{\dot{z}\dot{z}} - \frac{\Delta}{a^2\Sigma\times\overline{1-w}^{2}}$; or $\Sigma = 1 + \text{flu.}\frac{2\Pi\dot{z}}{\overline{1-w}^{3}}$, and $\frac{\ddot{w}}{\dot{z}\dot{z}} + w = 1 - \frac{\dot{\Sigma}}{2\Sigma}\times\frac{\dot{w}}{\dot{z}\dot{z}} - \frac{\Delta}{\Sigma\times\overline{1-w}^{2}}$, by making AC unity; which laſt equation will be rendered ſtill more commodious, by writing for $\dot{\Sigma}$ its equal $\frac{2\Pi\dot{z}}{\overline{1-w}^{3}}$; whence will be had $\frac{\ddot{w}}{\dot{z}\dot{z}} + w\left(= 1 - \frac{\Pi\dot{w}}{\Sigma\dot{z}\times\overline{1-w}^{3}} - \frac{\Delta}{\Sigma\times\overline{1-u}}\right) = \frac{1}{\Sigma}$ into $\Sigma - \Delta\times\overline{1-w}^{-2} - \Pi\times\frac{\dot{w}}{\dot{z}}\times\overline{1-w}^{-3}$: from which the values of w and Σ may be found, when thoſe of Δ and Π are aſſigned: by means whereof the time (t) of deſcribing the angle z, will alſo be known, from the equation $\frac{\dot{t}}{g} = \frac{x^2\dot{z}}{2aau}$ (above derived): which by ſubſtituting $c\,\Sigma^{\frac{1}{2}}$ and $\frac{1}{\overline{1-w}^{2}}$ for their equals u and $\frac{xx}{aa}$, gives $\dot{t} = \frac{g}{2c}\times\frac{\dot{z}}{\Sigma^{\frac{1}{2}}\times\overline{1-w}^{2}}$.

To exemplify the uſe of the equations here derived, by the reſolution of a caſe on which the determination of the lunar orbit depends, let the force Δ, whereby the body is ſolicited towards the center, be conſidered, as compoſed of two parts; whereof the principal ($b\times\overline{1-w}^{2}$) is in the inverſe duplicate-ratio of the diſtance; the other part, which is ſuppoſed ſmall in compariſon of the former, being as the diſtance $\left(\frac{1}{1-w}\right)$ directly, drawn into to a ſeries ($P'\times$ coſ. $pz+Q'\times$ coſ. $qz+R'\times$ coſ. rz, &c.) of coſines of multiples of the arch z, joined to ſmall, given, coefficients P', Q', R', &c. and let the force Π, acting in the perpendicular direction, be alſo ſuppoſed, as the diſtance directly, drawn into a ſeries ($P\times$ ſin. $pz+Q\times$ ſin. $qz+R\times$ ſin. rz * &c.) of ſines of multiples of the ſame arch, joined to ſmall, given, coefficients P, Q, R, &c. According to theſe aſſumptions, by ſubſtituting $b\times\overline{1-w}^{2}-\frac{1}{1-w}\times\overline{P'\text{coſ.}\,pz+Q'\,\text{coſ.}\,qz\ \&c.}$ and $\frac{1}{1-w}\times\overline{P\,\text{ſin.}\,pz+Q\,\text{ſin.}\,qz\ \&c.}$ for their equals Δ and Π, our two equations, $\Sigma=1+2$ fluent $\frac{\Pi\dot{z}}{\overline{1-w}^{3}}$, and $\frac{\ddot{w}}{\dot{z}\dot{z}}+w=\frac{1}{\Sigma}\times\overline{\Sigma-\Delta\times\overline{1-w}^{-2}-\Pi\times\frac{\dot{w}}{\dot{z}}\times\overline{1-w}^{-3}}$, will here become $\Sigma=1+2$ fluent $\overline{P\,\dot{z}\,\text{ſin.}\,pz+Q\,\dot{z}\,\text{ſin.}\,qz\ \&c.}\times\overline{1-w}^{-4}$, and $\frac{\ddot{w}}{\dot{z}\dot{z}}+w=\frac{1}{\Sigma}$ into $\Sigma-b-\overline{P'\text{coſ.}\,pz+Q'\,\text{coſ.}\,qz\ \&c.}\times\overline{1-w}^{-3}-\frac{\dot{w}}{\dot{z}}\times\overline{P\,\text{ſin.}\,pz+Q\,\text{ſin.}\,qz\ \&c.}\times\overline{1-w}^{-4}$.

Now, the orbit being ſuppoſed nearly circular, we may, in order to a firſt approximation, neglect w in both the factors $\overline{1-w}^{-3}$ and $\overline{1-w}^{-4}$, as being very ſmall in reſpect of unity; by which means Σ will become $=1+2$ flu. $p\dot{z}$ ſin. $pz+Q\dot{z}$ ſin. qz &c. $=1+d-\frac{2P}{p}\times$ coſ. $pz-\frac{2Q}{q}\times$ coſ. qz &c. (*ſee p.* 82.) where d, repreſenting the neceſſary correction to the fluent, muſt be taken $=\frac{2P}{p}+\frac{2Q}{q}$ &c. ſo that Σ may be $=1$, when $z=0$. This value of Σ being now ſubſtituted in the ſecond equation $\frac{\ddot{w}}{\dot{z}\dot{z}}+w=\frac{1}{\Sigma}\times\overline{\Sigma-b-P'\text{coſ.}\,pz-Q'\,\text{coſ.}\,qz\ \&c.}$ (where $\frac{\dot{w}}{\dot{z}}\times\overline{P\,\text{ſin.}\,pz+Q\,\text{ſin.}\,qz\ \&c.}$ on account of the ſmallneſs of $\frac{\dot{w}}{\dot{z}}$), there cometh out

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* *This aſſumption is not the leſs general by the multiples of* z *being taken the ſame here as in the value of* Δ; *becauſe, if any multiple of* z, *in the one value, enters not into the other, it is but ſuppoſing the correſponding coefficient in this laſt, to vaniſh or become equal to nothing.*

$$\frac{\ddot{w}}{\dot{z}\dot{z}}+w=\frac{1}{\Sigma}\times\overline{1+d-b-\overline{\frac{2P}{p}+P'}\times\text{coſ.}\,pz-\overline{\frac{2Q}{q}+Q'}\times\text{coſ.}\,qz\ \&c.}$$

where the general multiplicator $\frac{1}{\Sigma}$ may be alſo omitted, as differing very little from unity: this being done, and the fluent being taken, according to the method on *p.* 92, we thence find $w=1+d-b+\alpha\,\text{coſ.}\,z-\frac{P''\,\text{coſ.}\,pz}{1-pp}-\frac{Q''\,\text{coſ.}\,qz}{1-qq}$ &c. where $P''=\frac{2P}{p}+P'$, $Q''=\frac{2Q}{q}+Q'$, &c. and where the term α coſ. z, by which the fluent is corrected, muſt have its coefficient ſo taken, that w and z may have their origin together, that is, α muſt be made $=b-1-d+\frac{P''}{1-pp}+\frac{Q''}{1-qq}$ &c.

Having thus found a value nearly equal to w, we may by help thereof, proceed now to a ſecond approximation, by ſubſtituting that value for w, in the factors $\overline{1-w}|^{-3}$ and $\overline{1-w}|^{-4}$, wherein it was before neglected; and, to facilitate the computation, the terms in the value of $1-w$ (whoſe reciprocal is the diſtance of the body from the center of force) may be expreſſed by the general ſeries of coſines, $e\times\overline{1-B\,\text{coſ.}\,\beta z-C\,\text{coſ.}\,\gamma z-D\,\text{coſ.}\,\delta z\ \&c.}$ (as it appears from above, that the value of w will conſiſt of ſuch): by which means the ſame terms before determined will be again brought out, together with a number of others, ſerving as a farther correction. But, ſince the former operation is made, more with a view to diſcover the form of the ſeries, than to be regarded for its exactneſs, I ſhall have no farther reference thereto, but proceed to determine the value of the ſeveral quantities e, B, C &c. *de novo*, by a method ſomething different from that uſed above.

Firſt, then, from the equation $1-w=e\times\overline{1-B\,\text{coſ.}\,\beta z-C\,\text{coſ.}\,\gamma z-\ \&c.}$ will be had

$$\overline{1-w}|^{-3}=\frac{1}{e^3}+\frac{3}{e^3}\times\overline{B\,\text{coſ.}\,\beta z+C\,\text{coſ.}\,\gamma z\ \&c.}+\ \&c.$$

and $\overline{1-w}|^{-4} = \frac{1}{e^4} + \frac{4}{e^4} \times \overline{\mathrm{B}\ \text{coſ.}\ \beta z + \mathrm{C}\ \text{coſ.}\ \gamma z\ \&c.} + \&c.$ which laſt value being multiplied by $\mathrm{P}\dot{z} \times \text{ſin.}\ pz$ (according to the purport of our firſt equation, $\Sigma = 1 + 2$ fluent $\overline{\mathrm{P}\dot{z}\ \text{ſin.}\ pz + \mathrm{Q}\dot{z}\ \text{ſin.}\ qz\ \&c.} \times \overline{1-w}|^{-4}$), and a proper regard being, at the ſame time, had to the *Theorems on p.* 80, the product will ſtand thus,

$$\frac{\mathrm{P}\dot{z}}{e^4} \times \text{ſin.}\,pz + \frac{2\mathrm{P}\dot{z}}{e^4} \times \overline{-\mathrm{B}\,\text{ſin.}\,\overline{\beta - p}.\,z + \mathrm{B}\,\text{ſin.}\,\overline{\beta + p}.z\ \&c.}$$

or thus,

$$\frac{\mathrm{P}\dot{z}}{e^4} \times \text{ſin.}\,pz + \frac{2\mathrm{P}\dot{z}}{e^4} \times \overline{B\,\text{ſ.}\,\overline{p-\beta}.\,z + B\,\text{ſ.}\overline{p+\beta}.\,z + C\,\text{ſ.}\,\overline{p-\gamma}.z + C\,\text{ſ.}\,\overline{p+\gamma}.z\ \&c.}$$

whereof the fluent will be

$$-\frac{\mathrm{P}}{pe^4} \times \text{coſ.}\,pz - \frac{2\mathrm{P}}{e^4} \times \overline{\frac{B\text{coſ.}\overline{p-\beta}.z}{p-\beta} + \frac{B\text{coſ}\,\overline{p+\beta}.z}{p+\beta} + \frac{C\text{coſ.}\overline{p-\gamma}.z}{p-\gamma} + \frac{C\text{coſ.}\overline{p+\gamma}.z}{p+\gamma}\ \&c.}$$

In the very ſame manner, the terms ariſing from the multiplication of $\mathrm{Q}\dot{z} \times \text{ſin.}\ qz$ will be exhibited; and we ſhall therefore have

$$\Sigma = \begin{cases} 1 + d - \frac{2}{e^4} \times \overline{\frac{\mathrm{P}}{p} \times \text{coſ.}\,pz + \frac{\mathrm{Q}}{q} \times \text{coſ.}\,qz + \frac{\mathrm{R}}{r} \times \text{coſ.}\,rz\ \&c.} \\ - \frac{4\mathrm{P}}{e^4} \times \overline{\frac{B\,\text{coſ.}\,\overline{p-\beta}.z}{p-\beta} + \frac{B\text{coſ.}\,\overline{p+\beta}.z}{p+\beta} + \frac{C\text{coſ.}\,\overline{p-\gamma}.z}{p-\gamma} + \frac{C\,\text{coſ.}\overline{p+\gamma}.z}{p+\gamma}\ \&c.} \\ - \frac{4\mathrm{Q}}{e^4} \times \overline{\frac{B\,\text{coſ.}\,\overline{q-\beta}.z}{q-\beta} + \frac{B\text{coſ.}\,\overline{q+\beta}.z}{q+\beta} + \frac{C\,\text{coſ.}\,\overline{q-\gamma}.z}{q-\gamma} + \frac{C\,\text{coſ.}\,\overline{q+\gamma}.z}{q+\gamma}}. \\ \quad\&c. \qquad\qquad \&c. \end{cases}$$

In which the quantity *d*, aſſumed to denote the neceſſary correction, muſt be ſo taken, that Σ may be $=1$, when $z=0$, that is, *d* muſt be equal to $\frac{2}{e^4} \times \overline{\frac{\mathrm{P}}{p} + \frac{\mathrm{Q}}{q} + \frac{\mathrm{R}}{r}\ \&c.} +$ $\frac{4\mathrm{P}}{e^4} \times \overline{\frac{\mathrm{B}}{p-\beta} + \frac{\mathrm{B}}{p+\beta} + \frac{\mathrm{C}}{p-\gamma} + \frac{\mathrm{C}}{p+\gamma}\ \&c.} +$ $\frac{4\mathrm{Q}}{e^4} \times \overline{\frac{\mathrm{B}}{q-\beta} + \frac{\mathrm{B}}{q+\beta} + \frac{\mathrm{C}}{q-\gamma} + \frac{\mathrm{C}}{q+\gamma}\ \&c.} + \&c.$ Proceeding now to ſubſtitute in our other equation $\Big(\overline{\frac{\ddot{w}}{\dot{z}\dot{z}} + w} \times \Sigma - \Sigma + b + \overline{\mathrm{P}'\ \text{coſ.}\,pz + \mathrm{Q}'\ \text{coſ.}\,qz\ \&c.} \times \overline{1 + w}|^{-3} +$

$\frac{\dot{w}}{\dot{z}} \times \overline{P\,\text{ſin.}\,pz + Q\,\text{ſin.}\,qz\ \&c.} \times \overline{1-w}|^{-4} = 0$) we ſhall, in the firſt place, by taking the fluxion of $w = 1 - e + e \times \overline{B\,\text{coſ.}\,\beta z + C\,\text{coſ.}\,\gamma z\ \&c.}$ have $\frac{\dot{w}}{\dot{z}} = -e \times \overline{\beta B\,\text{ſin.}\,\beta z + \gamma C\,\text{ſin.}\,\gamma z\ \&c.}$ (*vid. p.* 82 *and* 83.) whereof the fluxion being, again, taken, we get $\frac{\ddot{w}}{\dot{z}\dot{z}} = -e \times \overline{\beta^2 B\,\text{ſin.}\,\beta z + \gamma^2 C\,\text{ſin.}\,\gamma z\ \&c.}$ and conſequently

$$\frac{\ddot{w}}{\dot{z}\dot{z}} + w = 1 - e + e \times \overline{\overline{1-\beta\beta} \times B\,\text{coſ.}\,\beta z + \overline{1-\gamma\gamma} \times C\,\text{coſ.}\,\gamma z\ \&c.}$$

Hence $\overline{\frac{\ddot{w}}{\dot{z}\dot{z}} + w} \times \Sigma - \Sigma = -e\Sigma +$

$$\overline{e \times \overline{1-\beta\beta} \times B\,\text{coſ.}\,\beta z + \overline{1-\gamma\gamma} \times C\,\text{coſ.}\,\gamma z\ \&c.} \times \overline{1 + d - \frac{2}{e^4} \times \overline{\frac{P}{p}\,\text{coſ.}\,pz + \frac{Q}{q} \times \text{coſ.}\,qz\ \&c.}}$$

which (by an actual multiplication of terms of the two ſeries's into each other, and a proper application of *Lem.* 1, *on p.* 76, neglecting at the ſame time, all terms wherein two, or more dimenſions of the quantities *B*, *C*, *D* &c. would ariſe) will be reduced to

$$\begin{array}{l}
-e\Sigma + \overline{1+d}.\overline{1-\beta\beta}.eB\,\text{coſ.}\,\beta z + \overline{1+d}.\overline{1-\gamma\gamma}.eC\,\text{coſ.}\,\gamma z + \overline{1+d}.\overline{1-\delta\delta}.eD\,\text{coſ.}\,\delta z\ \&c. \\
-\overline{1-\beta\beta}.\frac{B}{e^3} \times \overline{\frac{P}{p} \times \overline{\text{coſ.}\,\overline{p-\beta}.z + \text{coſ.}\,\overline{p+\beta}.z} + \frac{Q}{q} \times \overline{\text{coſ.}\,\overline{q-\beta}.z + \text{coſ.}\,\overline{q+\beta}.z}\ \&c.} \\
-\overline{1-\gamma\gamma}.\frac{C}{e^3} \times \overline{\frac{P}{p} \times \overline{\text{coſ.}\,\overline{p-\gamma}.z + \text{coſ.}\,\overline{p+\gamma}.z} + \frac{Q}{q} \times \overline{\text{coſ.}\,\overline{q-\gamma}.z + \text{coſ.}\,\overline{q+\gamma}.z}\ \&c.} \\
\quad \&c. \qquad\qquad \&c.
\end{array}$$

In like manner we have $\overline{P'\,\text{coſ.}\,pz + Q'\,\text{coſ.}\,qz\ \&c.} \times \overline{1-w}|^{-3} =$

$$\overline{P'\,\text{coſ.}\,pz + Q'\,\text{coſ.}\,qz\ \&c.} \times \overline{\frac{1}{e^3} + \frac{3}{e^3} \times \overline{B\,\text{coſ.}\,\beta z + C\,\text{coſ.}\,\gamma z\ \&c.}}$$

$$= \begin{cases}
\frac{1}{e^3} \times \overline{P'\,\text{coſ.}\,pz + Q'\,\text{coſ.}\,qz + R'\,\text{coſ.}\,rz\ \&c.} \\
+ \frac{3P'}{2e^3} \times \overline{B\,\text{coſ.}\,\overline{p-\beta}.z + B\,\text{coſ.}\,\overline{p+\beta}.z + C\,\text{coſ.}\,\overline{p-\gamma}.z + C\,\text{coſ.}\,\overline{p+\gamma}.z\ \&c.} \\
+ \frac{3Q'}{2e^3} \times \overline{B\,\text{coſ.}\,\overline{q-\beta}.z + B\,\text{coſ.}\,\overline{q+\beta}.z + C\,\text{coſ.}\,\overline{q-\gamma}.z + C\,\text{coſ.}\,\overline{q+\gamma}.z\ \&c.} \\
\quad \&c. \qquad\qquad \&c.
\end{cases}$$

Laſtly becauſe $\frac{\dot{w}}{\dot{z}} = -e \times \overline{\beta B\,\text{ſin.}\,\beta z + \gamma C\,\text{ſin.}\,\gamma z\ \&c.}$ it follows that $\frac{\dot{w}}{z}\times\overline{P\,\text{ſin.}\,pz+Q\,\text{ſin.}\,qz\ \&c.}\times\overline{1-w}|^{-4}=$ $-e\times\overline{\beta B\,\text{ſin.}\,\beta z+\gamma C\,\text{ſin.}\,\gamma z\ \&c.}\times\overline{P\,\text{ſin.}\,pz+Q\,\text{ſin.}\,qz\ \&c.}$ $\times\frac{1}{e^3}\times\frac{4}{e^3}\times\overline{B\,\text{coſ.}\,\beta z+C\,\text{coſ.}\,\gamma z\&c.}$ whence, by proceeding as above (neglecting the latter part, $\frac{4}{e^3}\times\overline{B\,\text{coſ.}\,\beta z+C\,\text{coſ.}\,\gamma z\ \&c.}$ of the laſt factor, as producing terms involving two dimenſions of the quantities B, C, D, &c.) will be had

$$-\frac{P}{e^3}\times\overline{\beta B\times\text{coſ.}\,\overline{p-\beta}.z-\beta B\,\text{coſ.}\,\overline{p+\beta}.z+\gamma C\,\text{coſ.}\,\overline{p-\gamma}.z-\gamma C\,\text{coſ.}\,\overline{p+\gamma}.z\ \&c.}$$

$$-\frac{Q}{e^3}\times\overline{\beta B\,\text{coſ.}\,\overline{q-\beta}.z-\beta B\,\text{coſ.}\,\overline{q+\beta}.z+\gamma C\,\text{coſ.}\,\overline{q-\gamma}.z-\gamma C\,\text{coſ.}\,\overline{q+\gamma}.z\ \&c.}$$

&c. &c.

Now let the three values, thus determined, be collected together, taking inſtead of Σ, its equal, as given by the firſt equation, putting $f=1+d$, and dividing the whole by $\overline{1+d}.e$; by which means our equation is, at length, reduced to

$$\left.\begin{array}{l}
\overline{1-\beta\beta}.\,B\,\text{coſ.}\,\beta z+\overline{1-\gamma\gamma}.\,C\,\text{coſ.}\,\gamma z+\overline{1-\delta\delta}.\,D\,\text{coſ.}\,\delta z\ \&c.\\
-1+\frac{b}{ef}+\frac{1}{e^4f}\times\overline{\frac{2P}{p}+P'}\times\text{coſ.}\,pz+\overline{\frac{2Q}{q}+Q'}\times\text{coſ.}\,qz\ \&c.\\
+\overline{\frac{4P}{p-\beta}+\frac{3P'}{2}-\frac{\beta P}{2}-\overline{1-\beta\beta}.\frac{P}{p}}\times\frac{B}{e^4f}.\ \text{coſ.}\,\overline{p-\beta}.z\\
+\overline{\frac{4P}{p+\beta}+\frac{3P'}{2}+\frac{\beta P}{2}-\overline{1-\beta\beta}.\frac{P}{p}}\times\frac{B}{e^4f}.\ \text{coſ.}\,\overline{p+\beta}.z\\
+\overline{\frac{4P}{p-\gamma}+\frac{3P'}{2}-\frac{\gamma P}{2}-\overline{1-\gamma\gamma}.\frac{P}{p}}\times\frac{C}{e^4f}.\ \text{coſ.}\,\overline{p-\gamma}.z\\
+\overline{\frac{4P}{p+\gamma}+\frac{3P'}{2}+\frac{\gamma P}{2}-\overline{1-\gamma\gamma}.\frac{P}{p}}\times\frac{C}{e^4f}.\ \text{coſ.}\,\overline{p+\gamma}.z\\
\&c.\quad \&c.
\end{array}\right\}=0$$

But as the coefficients of the terms in this equation are much compounded, it will be proper to make a ſubſtitution for them: Thus, let $\overline{P}=\overline{\frac{2}{p}+\frac{P'}{P}}\times\frac{P}{e^4f}$, $\overline{Q}=\overline{\frac{2}{q}+\frac{Q'}{Q}}\times\frac{Q}{e^4f}$ &c.

$$\overline{PB1}=\overline{\frac{4}{p-\beta}+\frac{3P'}{2P}-\frac{\beta}{2}-\frac{\overline{1-\beta\beta}}{p}}\times\frac{PB}{e^4f},$$

$$\overline{PB2}=\overline{\frac{4}{p+\beta}+\frac{3P'}{2P}+\frac{\beta}{2}-\frac{\overline{1-\beta\beta}}{p}}\times\frac{PB}{e^4f},$$

$$\overline{PC\,1} = \overline{\frac{4}{p-\gamma} + \frac{3P'}{2P} - \frac{\gamma}{2} - \frac{\overline{1-\gamma\gamma}}{p}} \times \frac{PC}{e^4 f},$$

$$\overline{PC\,2} = \overline{\frac{4}{p+\gamma} + \frac{3P'}{2P} + \frac{\gamma}{2} - \frac{1-\beta\beta}{p}} \times \frac{PC}{e^4 f}, \text{ \&c.}$$

ſuppoſing theſe ſubſtitutions to be continued on, to take in the terms affected by the other given quantities Q, q, R, r, &c. (which are had from thoſe above, by barely writing Q and q &c. in the place of P and p &c.) This being done, our equation will ſtand thus

$$\left.\begin{array}{llll}
\overline{1-\beta\beta}.\,B\,\text{coſ.}\,\beta z + \overline{1-\gamma\gamma}.\,C\,\text{coſ.}\,\gamma z + \overline{1-\delta\delta}.\,D\,\text{coſ.}\,\delta z & & & \\
\text{\&c.} & & & \\
-1+\frac{b}{ef} & +\overline{P}\times\text{coſ.}\,pz & & +\overline{Q}\times\text{coſ.}\,qz \text{ \&c.} \\
+\overline{PB1}\times\text{coſ.}\,\overline{p-\beta}.\,z & +\overline{PB2}\times\text{coſ.}\,\overline{p+\beta}.\,z & & \\
+\overline{QB1}\times\text{coſ.}\,\overline{q-\beta}.\,z & +\overline{QB2}\times\text{coſ.}\,\overline{q+\beta}.\,z \text{ \&c.} & & \\
+\overline{PC1}\times\text{coſ.}\,\overline{p-\gamma}.\,z & +\overline{PC2}\times\text{coſ.}\,\overline{p+\gamma}.\,z & & \\
+\overline{QC1}\times\text{coſ.}\,\overline{q-\gamma}.\,z & +\overline{QC2}\times\text{coſ.}\,\overline{q+\gamma}.\,z \text{ \&c.} & & \\
\text{\&c.} \qquad \text{\&c.} & & &
\end{array}\right\} = 0$$

From whence, by comparing the multiples of the arch z, in the firſt and ſecond lines, we have $\gamma=p$, $\delta=q$, and ſo on, to as many values (n) as there are quantities p, q, r, &c. And, by equating the correſponding coefficients of thoſe equi-multiples, we alſo have $C=-\frac{\overline{P}}{1-pp}$, $D=-\frac{\overline{Q}}{1-qq}$, $E=-\frac{\overline{R}}{1-rr}$, and ſo on, to n terms. Then, the value of the firſt n terms of the ſeries B coſ. $\beta z + C$ coſ. $\gamma z + D$ coſ. δz &c. (excluſive of B coſ. βz, of which more hereafter) being thus known, the terms in the 3d, 4th, 5th, &c. lines of our equation, being compounded of them and the given quantities P, p, Q, q, &c. will alſo become known; from whence, by continuing the compariſon with the terms of the upper line, after thoſe already taken, a new ſet of terms, involving two dimenſions of the quantities B, C, D, &c. P, P′, Q, Q′ &c. will be determined: by means whereof, ſtill continuing the operation in the ſame manner, a third ſet of terms may be obtained; and ſo on, at pleaſure.

As to the firſt term $\overline{1-\beta\beta}.\,B\,\text{coſ.}\,\beta z$, whereof no uſe has been yet made, it is reſerved to take off, or deſtroy any other term, or terms, of the ſame ſpecies, that may ariſe in the general equation. If no ſuch term ſhould occur, it is but making the coefficient $\overline{1-\beta\beta}.\,B=0$, and every thing will be right: But that ſuch terms do actually ariſe, will appear in the following illuſtration of the general method of proceeding, applied to a particular caſe, whereon the determination of the lunar orbit depends.

Let $P'=P$, $Q'=\frac{1}{3}P$, and Q, R &c. all equal to nothing *; then our equation will become

$$\left.\begin{array}{l}
\overline{1-\beta\beta}.B\,\text{coſ.}\,\beta z+\overline{1-\gamma\gamma}.C\,\text{coſ.}\,\gamma z+\overline{1-\delta\delta}.D\,\text{coſ.}\,\delta z+\overline{1-\varepsilon\varepsilon}.E\,\text{coſ.}\,\varepsilon z\ \&c.\\
-1+\frac{b}{ef}+\overline{P}\times\text{coſ.}\,pz+\overline{Q}+\overline{PB1}\times\text{coſ.}\,\overline{p-\beta}.z+\overline{PB2}\times\text{coſ.}\,\overline{p+\beta}.z\\
+\overline{\overline{QB1}+\overline{QB2}}\times\text{coſ.}\,\beta z+\overline{PC1}\times\text{coſ.}\,\overline{p-\gamma}.z+\overline{PC2}\times\text{coſ.}\,\overline{p+\gamma}.z\\
+\overline{\overline{QC1}+\overline{QC2}}\times\text{coſ.}\,\gamma z+\overline{PD1}\times\text{coſ.}\,\overline{p-\delta}.z+\overline{PD2}\times\text{coſ.}\,\overline{p+\delta}.z\\
+\overline{\overline{QD1}+\overline{QD2}}\times\text{coſ.}\,\delta z+\overline{PE1}\times\text{coſ.}\,\overline{p-\varepsilon}.z+\overline{PE2}\times\text{coſ.}\,\overline{p+\varepsilon}.z\\
+\overline{\overline{QE1}+\overline{QE2}}\times\text{coſ.}\,\varepsilon z+\ \&c.
\end{array}\right\}=0$$

Make now, $\gamma=p$, $\delta=p-\beta$, and $\varepsilon=p+\beta$; then the equation will ſtand thus

$$\left.\begin{array}{l}
\overline{1-\beta\beta}.\,B\,\text{coſ.}\,\beta z+\overline{1-\gamma\gamma}.\,C\,\text{coſ.}\,\gamma z+\overline{1-\delta\delta}.\,D\,\text{coſ.}\,\delta z+\ \&c.\\
-1+\frac{b}{ef}+\overline{P}\times\text{coſ.}\,pz+\overline{Q}+\overline{PB1}\times\text{coſ.}\,\overline{p-\beta}.z+\overline{PB2}\times\text{coſ.}\,\overline{p+\beta}.z\\
+\overline{\overline{QB1}+\overline{QB2}}\times\text{coſ.}\,\beta z+\overline{PC1}+\overline{PC2}\times\text{coſ.}\,2pz+\overline{\overline{QC1}+\overline{QC2}}\times\text{coſ.}\,pz\\
+\overline{PD1}\times\text{coſ.}\,\beta z+\overline{PD2}\times\text{coſ.}\,\overline{2p-\beta}.z+\overline{QD1}+\overline{QD2}\times\text{coſ.}\,\overline{p-\beta}.z,\ \&c.
\end{array}\right\}=0$$

Put $B'=\overline{QB1}+\overline{QB2}+\overline{PD1}+\ \&c.$

$C'=\overline{P}+\overline{QC1}+\overline{QC2}+\ \&c.$

$D'=\overline{PB1}+\overline{QD1}+\overline{QD2}+\ \&c.$

$E'=\overline{PB2}+\ \&c.$

$F'=\overline{PC2}+\ \&c.$

$G'=\overline{PD2}+\ \&c.$

Then, expunging the terms which have no multiple-coſine in them (as alſo deſtroying each other) we, at length have

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* That theſe aſſumptions are ſo made, as to expreſs (nearly) the forces whereby the ſun diſturbs the moon's motion about the earth, will be ſhewn hereafter.

$$\left.\begin{array}{llll}\overline{1-\beta\beta}.\,B\,\text{coſ.}\,\beta z+ & \overline{1-\gamma\gamma}.\,C\,\text{coſ.}\,\gamma z+ & \overline{1-\delta\delta}.\,D\,\text{coſ.}\,\delta z+ & \overline{1-\varepsilon\varepsilon}.\,E\,\text{coſ.}\,\varepsilon z\ \text{\&c.}\\ +B'\,\text{coſ.}\,\beta z+ & C'\text{coſ.}\,pz+ & D'\text{coſ.}\,\overline{p-\beta}.z+ & E'\text{coſ.}\,\overline{p+\beta}.z\\ +F'\,\text{coſ.}\,2pz+ & G'\text{coſ.}\,\overline{2p-\beta}.z+ & H'\text{coſ.}\,\overline{2p+\beta}.z+ & \text{\&c.}\end{array}\right\}=0$$

whence $\overline{1-\beta\beta}\times B=-B'$, $C=-\frac{C'}{1-pp}$, $D=-\frac{D'}{\overline{1-\overline{p-\beta}|^2}}$, $E=-\frac{E'}{\overline{1-\overline{p+\beta}|^2}}$, $F=-\frac{F'}{1-4pp}$, $G=-\frac{G'}{\overline{1-\overline{2p-\beta}|^2}}$, $H=-\frac{H'}{\overline{1-\overline{2p+\beta}|^2}}$, &c. and conſequently

$$1-w\ (=e\times\overline{1-B\,\text{coſ.}\,\beta z-C\,\text{coſ.}\,\gamma z\ \text{\&c.}})$$

$$=e\times\left\{\begin{array}{l}1-B\,\text{coſ.}\,\beta z+\frac{C'\times\text{coſ.}\,pz}{1-pp}+\frac{D'\,\text{coſ.}\,\overline{p-\beta}.\,z}{1-\overline{p-\beta}|^2}+\frac{E'\,\text{coſ.}\,\overline{p+\beta}.z}{1-\overline{p+\beta}|^2}\\ +\frac{F'\,\text{coſ.}\,2pz}{1-4pp}+\frac{G'\,\text{coſ.}\,\overline{2p-\beta}.z}{1-\overline{2p-\beta}|^2}+\frac{H'\text{coſ.}\,\overline{2p+\beta}.z}{1-\overline{2p+\beta}|^2}+\text{\&c.}\end{array}\right.$$

In deriving the equation here brought out, all terms involving two, or more dimenſions of the quantities B, C, D, &c. are neglected; it will, therefore, be neceſſary to ſhew now, how the effect of thoſe terms may be computed, and the approximation carried on, to any farther degree of exactneſs deſired. Previous to which, it will be proper to obſerve, that, in the preceding calculations, the quantity $\overline{1-w}|^{-3}$ was taken $=\frac{1}{e^3}+\frac{3}{e^3}\times\overline{B\,\text{coſ.}\,\beta z+C\,\text{coſ.}\,\gamma z\ \text{\&c.}}$ *barely*, and $\overline{1-w}|^{-4}=\frac{1}{e^4}+\frac{4}{e^4}\times\overline{B\,\text{coſ.}\,\beta z+C\,\text{coſ.}\,\gamma z\ \text{\&c.}}$ whereas the true value of $\overline{1-w}|^{-3}$, or $e\times\overline{1-B\,\text{coſ.}\,\beta z-C\,\text{coſ.}\,\gamma z\text{\&c.}}|^{-3}$ is $=\frac{1}{e^2}+\frac{3}{e^3}\times\overline{B\,\text{coſ.}\,\beta z+C\,\text{coſ.}\,\gamma z\ \text{\&c.}}+\frac{6}{e^3}\times\overline{B\,\text{coſ.}\,\beta z+C\,\text{coſ.}\,\gamma z\text{\&c.}}|^2+\text{\&c.}$ and that of $\overline{1-w}|^{-4}=\frac{1}{e^4}+\frac{4}{e^4}\times\overline{B\,\text{coſ.}\,\beta z+C\,\text{coſ.}\,\gamma z\ \text{\&c.}}$ $+\frac{10}{e^4}\times\overline{B\,\text{coſ.}\,\beta z+C\,\text{coſ.}\,\gamma z\text{\&c.}}|^2$ Now $\overline{B\,\text{coſ.}\,\beta z+C\,\text{coſ.}\,\gamma z\text{\&c.}}|^2$ (before neglected) is evidently equal to $B^2\times\overline{\text{coſ.}\,\beta z}|^2+2\,B\,C\times\text{coſ.}\,\beta z\times C\,\text{coſ.}\,\gamma z\ \text{\&c.}=\frac{1}{2}B^2\times\overline{1+\text{coſ.}\,2\beta z}+\overline{BC\times\text{coſ.}\,\overline{\gamma-\beta}.\,z+\text{coſ.}\,\overline{\gamma+\beta}.\,z}+\text{\&c.}$ all which coſines, together with their coefficients, will be known, ſeeing β, B, γ, C &

&c. are given (or nearly ſo) by the preceding operation. And, in the ſame manner, a ſeries of coſines expreſſing the value of $\overline{\text{B coſ. }\beta z+\text{C coſ. }\gamma z\ \&\text{c.}}|^{3}$ (or of the moſt conſiderable terms thereof) may be found, ſhould it be neceſſary to continue the approximation ſo far.

To find what alteration theſe quantities will produce in the value of $1-w$ (before found) let any new term thus ariſing in the value of $\overline{1-w}|^{-3}$, be denoted by $\frac{\text{M}}{e^3}\times\text{coſ. }\alpha z$, and the term anſwering to it in the value of $\overline{1-w}|^{-4}$, by $\frac{\text{N}}{e^4}\times\text{coſ. }\alpha z$: then the correſponding increaſe in the value of $-e\Sigma$ $(=-e-2e\times\text{flu. }\overline{\text{P}\dot{z}\text{ ſin. }pz+\text{Q}\dot{z}\text{ ſin. }qz.\ \&\text{c.}}\times\overline{1-w}|^{-4},)$ will be $-2e\times\text{flu. }\overline{\text{P}\dot{z}\text{ ſin. }pz+\text{Q}\dot{z}\text{ ſin. }qz.\ \&\text{c.}}\times\frac{\text{N}}{e^4}\times\text{coſ. }\alpha z=-\frac{\text{N}}{e^3}$ into flu. $\text{P}\dot{z}\times\overline{\text{ſin. }\overline{p-\alpha}.z+\text{ſin. }\overline{p+\alpha}.z}+\text{Q}\dot{z}\times\overline{\text{ſin. }\overline{q-\alpha}.z+\text{ſin. }\overline{q+\alpha}.z}\ \&\text{c.}$

$$=\frac{\text{N}}{e^3}\text{ into }\frac{\text{P coſ. }\overline{p-\alpha}.z}{p-\alpha}+\frac{\text{P coſ. }\overline{p+\alpha}.z}{p+\alpha}+\frac{\text{Q coſ. }\overline{q-\alpha}.z}{q-\alpha}+\frac{\text{Q coſ. }\overline{q+\alpha}.z}{q+\alpha}$$

&c.

In like manner, the increaſe of the ſecond member $(\overline{\text{P}'\text{ coſ. }pz+\text{Q}'\text{ coſ. }qz\ \&\text{c.}}\times\overline{1-w}|^{-3})$ of our general equation (*ſee p.*147.) appears to be $\overline{\text{P}'\text{ coſ. }pz+\text{Q}'\text{ coſ. }qz\ \&\text{c.}}\times\frac{\text{M}}{e^3}\times\text{coſ. }\alpha z$

$$=\frac{\text{M}}{2e^3}\times\overline{\text{P}'\text{ coſ. }\overline{p-\alpha}.z+\text{P}'\text{ coſ. }\overline{p+\alpha}.z+\text{Q}'\text{ coſ. }\overline{q-\alpha}.z+\text{Q}'\text{ coſ. }\overline{q+\alpha}.z}\ \&\text{c.}$$

And the increaſe of the laſt term, or member, $\overline{\text{P ſin. }pz+\text{Q ſin. }qz\ \&\text{c.}}\times\frac{1}{3\dot{z}}\times\text{flux. }\overline{1-w}|^{-3}$, is had $=$

$$\overline{\text{P ſin. }pz+\text{Q ſin. }qz\ \&\text{c.}}\times\frac{1}{3\dot{z}}\times-\frac{\text{M}\alpha\dot{z}}{e^3}\times\text{ſin. }\alpha z$$

$$=\overline{\text{P ſin. }pz+\text{Q ſin. }qz\ \&\text{c.}}\times-\frac{\text{M}\alpha}{3e^3}\times\text{ſin. }\alpha z=$$

$$-\frac{\alpha\text{M}}{6e^3}\times\overline{\text{P coſ. }\overline{p-\alpha}.z-\text{P coſ. }\overline{p+\alpha}.z+\text{Q coſ. }\overline{q-\alpha}.z-\text{Q coſ. }\overline{q+\alpha}.z\ \&\text{c.}}$$

Let theſe three quantities be now collected together; from whence the new terms entering into the general equation, when the whole is divided by *ef*, will ſtand thus

$$\frac{P}{e^4f}\times\overline{\frac{N}{p-\alpha}+\frac{MP'}{2P}-\frac{\alpha M}{6}}\times\text{coſ.}\,\overline{p-\alpha}.z+\frac{P}{e^4f}\times\overline{\frac{N}{p+\alpha}+\frac{MP'}{2P}+\frac{\alpha M}{6}}\times\text{coſ.}\,\overline{p+\alpha}.z$$

$+$ &c.

So that, from the general method of operation before laid down, it appears that the proper correction for the value of $1-w$, ariſing from the terms, $\frac{M}{e^3}\times\text{coſ.}\,\alpha z$ and $\frac{N}{e^4}\times\text{coſ.}\,\alpha z$, before neglected in the reſpective values of $\overline{1-w}|^{-3}$ and $\overline{1-w}|^{-4}$, will be expreſſed by *e into* $\frac{P}{e^4f}\times\overline{\frac{N}{p-\alpha}+\frac{MP'}{2P}-\frac{\alpha M}{6}}\times\frac{\text{coſ.}\,\overline{p-\alpha}.\,z}{1-\overline{p-\alpha}|^2}+$

$$\frac{P}{e^4f}\times\overline{\frac{N}{p-\alpha}+\frac{MP'}{2P}+\frac{\alpha M}{6}}\times\frac{\text{coſ.}\,\overline{p+\alpha}.z}{1-\overline{p+\alpha}|^2}+$$

$$\frac{Q}{e^4f}\times\overline{\frac{N}{q-\alpha}+\frac{MQ'}{2Q}-\frac{\alpha M}{6}}\times\frac{\text{coſ.}\,\overline{q-\alpha}.\,z}{1-\overline{q-\alpha}|^2}+$$

$$\frac{Q}{e^4f}\times\overline{\frac{N}{q+\alpha}+\frac{MQ'}{2Q}+\frac{\alpha M}{6}}\times\frac{\text{coſ.}\,\overline{q+\alpha}.\,z}{1-\overline{q+\alpha}|^2}+\&c.$$

In reſpect to which it may be obſerved, that no regard has been had to ſuch quantities as would ariſe from the multiplication of the new terms in Σ, by the ſeries $\overline{1-\beta\beta}.\,B\,\text{coſ.}\,\beta z+\overline{1-\gamma\gamma}.\,C\,\text{coſ.}\,\gamma z+\&c.$ (as ought, in ſtrictneſs, to have been done); becauſe theſe terms being very ſmall, they will, after multiplication into the ſmall quantities $\overline{1-\beta\beta}\times B$, $\overline{1-\gamma\gamma}\times C$ &c. be ſo far reduced, as to become quite inconſiderable. And it may be obſerved farther, that even the whole product ariſing from the multiplication of the value of Σ into the ſaid ſeries, except the terms $-e\Sigma+\overline{1-\beta\beta}.efB\,\text{coſ.}\,\beta z+\overline{1-\gamma\gamma}.efC\,\text{coſ.}\,\gamma z$ &c. might be alſo rejected, without producing an error of more than a few ſeconds, in finding the place of the moon in her orbit.

Having ſhewn what alteration will be cauſed in the value of $1-w$ (the reciprocal of the diſtance of the body from the center of force) from the taking in of any ſmall, aſſigned terms in the values of $\overline{1-w}|^{-3}$ and $\overline{1-w}|^{-4}$, it may be a proper place here to ſhew (as it is of great importance to be known) what change will ariſe in the ſaid value of $1-w$ from the addition of any ſmall, new terms, $A\,\text{ſin.}\,\pi z$, and A′ coſ. πz, to the reſpective forces P ſin. pz + Q ſin. qz &c. and P′ coſ. pz + Q′ coſ. qz &c. whereby the motion of the body is diſturbed. What this alteration, or correction ought to be, is eaſily diſcovered from the general equation on *p.* 151; from whence, by ſubſtituting π, A, and A′ inſtead of p, P, and P′, reſpectively, the new terms affected by π, entering into the ſaid equation, will appear to be

$$\frac{A}{e^4 f} \times \overline{\frac{2}{\pi} + \frac{A'}{A}} \times \text{coſ.}\ \pi z$$

$$+ \overline{\frac{4}{\pi-\beta} + \frac{3A'}{2A} - \frac{\beta}{2} - \frac{1-\beta\beta}{\pi}} \times \frac{AB}{e^4 f} \times \text{coſ.}\ \overline{\pi-\beta}.\, z$$

$$+ \overline{\frac{4}{\pi+\beta} + \frac{3A'}{2A} + \frac{\beta}{2} - \frac{1-\beta\beta}{\pi}} \times \frac{AB}{e^4 f} \times \text{coſ.}\ \overline{\pi+\beta}.\, z.\ \text{\&c.}$$

and conſequently the increaſe in the value of $1-w$ ariſing therefrom = e *into* $\frac{A}{e^4 f} \times \overline{\frac{2}{\pi} + \frac{A'}{A}} \times \frac{\text{coſ.}\ \pi z}{1-\pi\pi}$ +

$$\overline{\frac{4}{\pi-\beta} + \frac{3A'}{2A} - \frac{\beta}{2} - \frac{1-\beta\beta}{\pi}} \times \frac{AB}{e^4 f} \times \frac{\text{coſ.}\ \overline{\pi-\beta}.\, z}{1-\overline{\pi-\beta}\,\rvert^2} +$$

$$\overline{\frac{4}{\pi+\beta} + \frac{3A'}{2A} + \frac{\beta}{2} - \frac{1-\beta\beta}{\pi}} \times \frac{AB}{e^4 f} \times \frac{\text{coſ.}\ \overline{\pi+\beta}.\, z}{1-\overline{\pi+\beta}\,\rvert^2} + \text{\&c.}$$

where, in many caſes, the firſt term alone, will be ſufficient.

In making the different corrections above pointed out, it will be found neceſſary to have a particular regard to ſuch *multiples* of the arch z, as are, either, very ſmall, or nearly equal to *unity* (of which two kinds, thoſe whoſe exponents are $p-2\beta$, and $p-\beta$, will be found the moſt conſiderable.) For, though the coefficients of *ſuch* terms ſhould appear, at firſt, to be ſmall, they ought not, therefore, to be immediately rejected; becauſe the diviſors which they afterwards receive (the former in obtaining the value of $1-w$, and the latter, in finding the *anomaly* from thence) are ſuch as may render the effect, or quotient, too conſiderable to be intirely diſregarded.

And it may be eaſily conceived, without the help of calculation, that a term, or force of the former kind, expreſſed by the *ſine* or *co-ſine* of a very ſmall multiple of the longitude z, muſt neceſſarily have a much greater effect, than another (having the ſame coefficient) which is proportional to the ſine or co-ſine of a *large multiple* of the ſame angle: becauſe, when the *index*, or *multiple* is a very ſmall one, the term itſelf, while z increaſes, will continue, for a conſiderable time, nearly of the ſame value; and conſequently, will have its whole effect exerted in the ſame direction; but when the *multiple* is a large one, the changes from *poſitive* to *negative*, and from thence to *poſitive* again, are ſo quick, that ſufficient time is not allowed for producing any conſiderable inequality in the body's motion, before that effect is again deſtroyed by the ſame force, acting equally in the oppoſite direction.

The value of $1-w$ (the reciprocal of the body's diſtance from the center of force) being, by the *formulæ* laid down above, approximated to a ſufficient degree of exactneſs, we may from thence, and the equation $\dot{t} = \frac{g}{2c} \times \frac{\dot{z}}{\Sigma^{\frac{1}{2}} \times \overline{1-w}|^{2}}$ (given at *page* 146.) proceed to compute the time (t) of deſcribing the angle z; whereby the difference between the *true* and *mean* anomalies will alſo be known; which, in the *lunar theory*, is the great point in queſtion, and is beſides, abſolutely neceſſary in order to introduce the proper quantities of the forces whereby the moon's motion is diſturbed.

Let, therefore, the value of Σ (as given by the firſt equation, *at p.* 149,) be here repreſented by
$f \times \overline{1-\bar{B}\,\text{coſ.}\,\beta z - C\,\text{coſ.}\,\gamma z - \bar{D}\,\text{coſ.}\,\delta z\ \&c.}$ ſo ſhall

$$\dot{t} = \frac{g}{2c} \times f^{\frac{1}{2}} \times \overline{1-\bar{B}\,\text{coſ.}\,\beta z - \bar{C}\,\text{coſ.}\,\gamma z\ \&c.}|^{-\frac{1}{2}} \times e^{-2} \times \overline{1 - B\ \text{coſ.}\ \beta z\ \&c.}|^{-2}$$

But $\overline{1-\bar{B}\,\text{coſ.}\,\beta z - \bar{C}\,\text{coſ.}\,\gamma z\&c.}|^{-\frac{1}{2}} = 1 + \frac{1}{2} \times \overline{\bar{B}\ \text{coſ.}\ \beta z\ \&c.} +$
$\frac{3}{8} \times \overline{\bar{B}\,\text{coſ.}\,\beta z + \bar{C}\,\text{coſ.}\,\gamma z\ \&c.}|^{2} = 1 + \frac{1}{2} \times \overline{\bar{B}\,\text{coſ.}\,\beta z + \bar{C}\,\text{coſ.}\,\gamma z\ \&c.} +$
$\frac{3}{8} \times \frac{1}{2}\,\bar{B}\bar{B} \times \overline{1 + \text{coſ.}\,2\beta z} + BC \times \overline{\text{coſ.}\,\overline{\gamma-\beta}.z + \text{coſ.}\,\overline{\gamma+\beta}.z\ \&c.}$
And $\overline{1-B\,\text{coſ.}\,\beta z - C\,\text{coſ.}\,\gamma z\&c.}|^{-2} = 1 + 2 \times \overline{B\ \text{coſ.}\ \beta z\ \&c.} +$
$3 \times \frac{1}{2}\,BB \times \overline{1 + \text{coſ.}\,2\beta z} + BC \times \overline{\text{coſ.}\,\overline{\gamma-\beta}.z + \text{coſ.}\,\overline{\gamma+\beta}.z\ \&c.}$
Which values being multiplied together, we thence get $\dot{t} =$

$$\frac{g\dot{z}}{2ceef^{\frac{1}{2}}} \text{ into } 1 + \begin{cases} 2 \times \overline{B\,\text{coſ.}\,\beta z + C\,\text{coſ.}\,\gamma z\ \&c.} \\ \frac{1}{2} \times \overline{\bar{B}\ \text{coſ.}\ \beta z + \bar{C}\ \text{coſ.}\ \gamma\ z\ \&c.} \end{cases}$$

$$+\left\{\begin{array}{l} 3\times\tfrac{1}{2}BB\times\overline{1+\cos.2\beta z}+BC\times\overline{\cos.\overline{\gamma-\beta}.z+\cos.\overline{\gamma+\beta}.z}\ \&c. \\ \tfrac{3}{8}\times\tfrac{1}{2}\bar{B}\bar{B}\times\overline{1+\cos.2\beta z}+\bar{B}\bar{C}\times\overline{\cos.\overline{\gamma-\beta}.z+\cos.\overline{\gamma+\beta}.z}\ \&c. \\ \tfrac{1}{2}\times B\bar{B}\times\overline{1+\cos.2\beta z}+\overline{B\bar{C}+B\bar{C}}\times\overline{\cos.\overline{\gamma-\beta}.z+c.\overline{\gamma+\beta}.z}\ \&c. \end{array}\right.$$

Put $h=1+\tfrac{3}{2}\times\overline{BB+CC+DD}\ \&c.+\tfrac{3}{16}\times\overline{\bar{B}\bar{B}+\bar{C}\bar{C}+\bar{D}\bar{D}}\ \&c.+\tfrac{1}{2}\times\overline{B\bar{B}+C\bar{C}+D\bar{D}}\ \&c.$ $(B)=2B+\tfrac{1}{2}\bar{B}$, $(BB)=\tfrac{3}{2}BB+\tfrac{3}{16}\bar{B}\bar{B}+\tfrac{1}{2}B\bar{B}$, $(BC)=3BC+\tfrac{3}{8}\bar{B}\bar{C}+\tfrac{1}{2}\bar{B}C+\tfrac{1}{2}B\bar{C}$, &c. continuing on theſe ſubſtitutions, ſo as to take in the terms affected by the other quantities D, E, F, &c. (which are had from thoſe above, by barely writing one letter for another); by means whereof our equation is reduced to

$$\dot{t}=\frac{g\dot{z}}{2ce^2f^{\frac{1}{2}}}\times\left\{\begin{array}{l} h+(B)\times\cos.\beta z+(C)\times\cos.\gamma z+(D)\times\cos.\delta z\ \&c. \\ (BB)\times\cos.2\beta z+(BC)\times\overline{\cos.\overline{\gamma-\beta}.z+\cos.\overline{\gamma+\beta}.z} \\ (CC)\times\cos.2\gamma z+(CD)\times\overline{\cos.\overline{\gamma-\delta}.z+\cos.\overline{\gamma+\delta}.z} \\ \&c. \qquad\qquad\qquad \&c. \end{array}\right.$$

From whence, by taking the fluent &c. we have

$$t=\frac{gh}{2ce^2f^{\frac{1}{2}}}\times\left\{\begin{array}{l} z+\dfrac{(B)\times\text{ſin}.\beta z}{h\beta}+\dfrac{(C)\times\text{ſin}.\gamma z}{h\gamma}+\dfrac{(D)\times\text{ſin}.\delta z}{h\delta}+\&c. \\ \dfrac{(BB)\times\text{ſin}.2\beta z}{2h\beta}+\dfrac{(BC)}{h}\times\overline{\dfrac{\text{ſin}.\overline{\gamma-\beta}.z}{\gamma-\beta}+\dfrac{\text{ſin}.\overline{\gamma+\beta}.z}{\gamma+\beta}}+\&c. \\ \dfrac{(CC)\times\text{ſin}.2\gamma z}{2h\gamma}+\dfrac{(CD)}{h}\times\overline{\dfrac{\text{ſin}.\overline{\gamma-\delta}.z}{\gamma-\delta}+\dfrac{\text{ſin}.\overline{\gamma+\delta}.z}{\gamma+\delta}}+\&c. \end{array}\right.$$

In reſpect to which it will be needful to obſerve, that the firſt term, $\frac{gh}{2ce^2f^{\frac{1}{2}}}\times z$, will, when $z=$ the whole circumference, be the true meaſure of the mean periodic time; becauſe all the other terms being compoſed of the ſines of multiples of the arch z, they will, while z keeps increaſing, change from *poſitive* to *negative*, and from *thence* to *poſitive* again, and ſo on continually; and therefore can have nothing to do in the *mean motion*; being themſelves no other than the proper equations whereby the *mean* and *true* motions differ from each other; ſo that, the *true motion* being defined by z, the *mean motion* will be expreſſed by $z+\frac{(B)\,\text{ſin}.\beta z}{h\beta}+\frac{(C)\,\text{ſin}.\gamma z}{h\gamma}$ &c. as above determined.

From

Fig. 31. From the expreſſion $\frac{gh}{2ce^2 f^{\frac{1}{2}}} \times z$, here found, the proportion between the mean periodic time of the body in its orbit ABP &c. and the periodic time in the circular orbit ADK, that might be deſcribed, independent of the perturbating forces, by means of a centripetal force ſufficient to cauſe the body to move therein, will be known: for the quantities e, f, and h being here, each, equal to unity, the ſaid expreſſion will, in this caſe, become $= \frac{g}{2c} \times z$: whence it is evident, that the periodic time in the circle, will be in proportion to the periodic time in the orbit ABP &c. as *unity* is to $\frac{h}{e^2 f^{\frac{1}{2}}}$.

APPLICATION to the *Lunar Orbit*.

In order to apply the concluſions derived in the preceding pages, to the determination of the lunar orbit and the different *inequalities* of the motion therein, it will be neceſſary, firſt of all, to inveſtigate the ſun's force to diſturb the motion of the moon about the center of the earth; from whence all thoſe *inequalities*, except *that* ariſing from the excentricity, are produced.

Fig. 34. Let C, S, and B repreſent any three cotemporary places of the earth, ſun, and moon, reſpectively; and, upon the diagonal BS, let the parallelogram BCSH be conſtituted; making BF perpendicular to CS.

If k be aſſumed to denote accelerative force of the earth to the ſun, the accelerative force of the moon to the ſun will be truly repreſented by $k \times \frac{\overline{SC}^2}{\overline{SB}|^2}$; which force may be reſolved into two others, the one in the direction BC, expreſſed by $k \times \frac{\overline{SC}^2}{\overline{SB}|^2} \times \frac{BC}{SC}$; and the other in the direction BH, expreſſed by $k \times \frac{\overline{SC}^2}{\overline{SB}|^2} \times \frac{SC}{SB}$; from which laſt, let the force k in the parallel direction CS be ſubtracted; ſo ſhall the remainder

$k \times \frac{\overline{SC}^3 - \overline{SB}^3}{\overline{SB}^3}$ be that part of the force, acting in the direction BH, whereby the motion of the moon about the earth is diſturbed: But this quantity, $k \times \frac{\overline{SC}^3 - \overline{SB}^3}{\overline{SB}^3}$, is evidently equal to $k \times \overline{SC - SB} \times \frac{\overline{SC}^2 + SC \times SB + \overline{SB}^2}{\overline{SB}^3}$; which, as SB (by reaſon of the great diſtance of the ſun) is nearly equal to SF, will be alſo equal to $k \times CF \times \frac{\overline{SC}^2 + SC \times SB + \overline{SB}^2}{\overline{SB}^3}$, or to $3k \times \frac{CF}{SC}$, very near; becauſe SC being nearly equal to SB, it may be here ſubſtituted inſtead thereof. Now this force $3k \times \frac{CF}{SC}$, acting in the direction BH, parallel to SC, may be, again, reſolved into two others; the one in a direction perpendicular to BC, expreſſed by $3k \times \frac{CF}{SC} \times \text{ſin.}\, SCB = 3k \times \frac{CB}{SC} \times \text{ſin.}\, SCB \times \text{coſ.}\, SCB = \frac{3k}{2} \times \frac{CB}{CS} \times \text{ſin.}\, 2SCB$; and the other in the direction of CB, expreſſed by $3K \times \frac{CF}{SC} \times \text{coſ.}\, SCB = 3k \times \frac{CB}{SC} \times \overline{\text{coſ.}SCB}^2 = \frac{3k}{2} \times \frac{CB}{SC} \times \overline{1 + \text{coſ.}\, 2SCB}$: from which laſt, let the force $k \times \frac{\overline{SC}^2}{\overline{SB}^2} \times \frac{BC}{SB}$ (or $k \times \frac{BC}{SC}$) above found, acting in the oppoſite direction BC, be ſubtracted, ſo ſhall the remainder $k \times \frac{CB}{SC} \times \overline{\tfrac{1}{2} + \tfrac{3}{2}\, \text{coſ.}\, 2SCB}$ be the diminution of the centripetal force to the earth, ariſing from the action of the ſun.

To exterminate k and CS from theſe expreſſions, let the given quantity 0,0748, expreſſing the mean periodic time of the moon, in parts of an year (*as found by obſervation*) be denoted by m; then (*by the proportion on p.* 160,) as $\frac{b}{e^2 f^{\frac{1}{2}}}$ is to 1, ſo is m to $\left(\frac{mc^2 f^{\frac{1}{2}}}{b}\right)$ the periodic time in the circular orbit ADK, that might be deſcribed, independent of the perturbating forces, by means of a centripetal force ſufficient to cauſe the moon to revolve therein. But in circles the centripetal forces are known to be as the radii directly, and the ſquares of the periodic times inverſely; whence we have, as 1 (the force in the circle ADK) is to *k*, the mean force whereby the earth is retained in its orbit about the ſun, ſo is $\frac{\mathrm{CA}}{\frac{\overline{me^2f^{\frac{1}{2}}}\rvert^{2}}{hh}}$ to $\frac{\mathrm{CS}}{1^2}$: from which proportion $\frac{k}{\mathrm{CS}}=\frac{m^2e^4f}{hh\times\mathrm{CA}}$\*: And the forces $k\times\frac{\mathrm{CB}}{\mathrm{CS}}\times\overline{\tfrac{1}{2}+\tfrac{1}{2}\,\mathrm{cos.}\,2\mathrm{SCB}}$, and $\frac{k\times\mathrm{CB}}{\mathrm{CS}}\times\frac{1}{2}$ ſin. 2 SCB, whereby the motion of the moon about the earth is diſturbed, will therefore be truly defined by $\frac{3m^2e^4f}{2hh}\times\frac{\mathrm{CB}}{\mathrm{CA}}\times\overline{\mathrm{cos.}\,2\mathrm{SCB}+\tfrac{1}{3}}$, and $\frac{3m^2e^4f}{2hh}\times\frac{\mathrm{CB}}{\mathrm{CA}}\times$ ſin. 2 SCB.

If the ſun and moon be ſuppoſed to move from the line CA at the ſame time, ſo that the angle ACS may be the ſun's apparent motion about the earth, whilſt the moon in her orbit moves from A to B, it will be, as $1:m::z$ (= the angle ACB): mz = the angle ACS, *nearly*; which would be ſtrictly true, were the *true motions* of the ſun and moon to be exactly in the ſame proportion with the *mean motions*. Hence SCB (=ACB—ACS) will be had $=z-mz$, nearly; and conſequently $2\mathrm{SCB}=pz$ (by making $p=\overline{1-m}\times 2$): and ſo the forces found above, by ſubſtituting this value, and writing $\frac{1}{1-w}$ in the room of its equal $\frac{\mathrm{CB}}{\mathrm{CA}}$, will, by means thereof, be reduced to $\frac{1}{1-w}\times\overline{\frac{3m^2e^4f}{2hh}\times\mathrm{cof.}\,pz+\frac{m^2e^4f}{2hh}}$, and $\frac{1}{1-w}\times\frac{3m^2e^4f}{2hh}\times\mathrm{ſin.}\,pz$: which quantities, with contrary ſigns (becauſe they diminiſh the centripetal force to the earth, and the area deſcribed, inſtead of increaſing them) being compared with the two general expreſſions $\frac{1}{1-w}\times\overline{\mathrm{P}'\,\mathrm{cof.}\,pz+\mathrm{Q}'\,\mathrm{cof.}\,qz+\mathrm{R}'\,\mathrm{cof.}\,rz\ \&c.}$ and $\frac{1}{1-w}\times\overline{\mathrm{P}\,\mathrm{ſin.}\,pz+\mathrm{Q}\,\mathrm{ſin.}\,qz+\mathrm{R}\,\mathrm{ſin}\,rz\ \&c.}$ as given on *p.* 147, we ſhall, in this caſe, have $q = 0$, $r = 0$, &c. $P' = -\frac{3m^2}{2} \times \frac{e^4 f}{bb} = -0{,}0084 \times \frac{e^4 f}{bb}$, $Q = -\frac{m^2}{2} \times \frac{e^4 f}{bb} = -0{,}0028 \times \frac{e^4 f}{bb}$ $(= \frac{1}{3} P')$, $P = -\frac{3m^2}{2} \times \frac{e^4 f}{bb}$ $(= P')$, $Q = 0$, $R = 0$ &c. But here, inſtead of b, or its equal, $1 + \frac{3}{2} \times \overline{BB + CC + DD}$ &c. (given at *p.* 159,) it will be ſufficient to make uſe of the firſt term of the ſeries only (till a more exact value, by means of the ſubſequent calculations, can be known); becauſe all the quantities B, C, D, &c. being ſmall in compariſon of *unity*, the ſquares of them, which are here neglected, will be ſtill ſmaller, and of leſs conſequence.

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* *In deriving this concluſion, regard is had to the moon's relative motion about the earth's center; but if we conſider the motion, as performed about the common center of gravity of the earth and moon, the reſult will be exactly the ſame.*

If, now, the caſe under conſideration be compared with *that* laid down, and reſolved, at *p.* 153, they will appear to be the ſame; ſo that we have nothing more to do here, than to compute, in numbers, the different values of the algebraic quantities there brought out. But one thing previous thereto muſt be taken notice of, reſpecting the principal (B coſ. βz) of thoſe values, on which the great elliptical equation, ariſing from the eccentricity, depends; which cannot be known but from the obſervations of *Aſtronomers*; ſince it is owing to the projectile-velocity which the moon, *firſt*, received, more than to the perturbating force of the ſun, whoſe effect we are about to calculate. But, though the term $B \times$ coſ. βz, cannot be determined by *theory* alone, yet the value of its exponent β, on which the motion of the *apogee* depends, may from hence be deduced. This value, in a former operation (ſee *page* 148,) was found to be an *unit* (as the circumſtance of the problem, when the *apogee* is at reſt, abſolutely requires); the true value cannot, therefore, differ much from an *unit*, which may be uſed inſtead of β, till a more exact value, by means of a ſecond operation, is known. Making, therefore, $\beta = 1$; $\frac{P}{e^4 f} = -0{,}0084$; $P' = P$; $Q = 0$; $Q' = \frac{1}{3} P' = \frac{1}{3} P$; $m = 0{,}0748$, p $(= 1 - m \times 2) = 1{,}8504$ $\gamma = p$, $\delta = p - \beta$, $\varepsilon = p + \beta$, &c. (as before determined) and ſubſtituting in the equations on *p.* 151, 152, 153, we have

$\overline{P}\left(=\frac{2P}{pe^4f}+\frac{P'}{e^4f}\right)=\frac{2x-0{,}0084}{p}-0{,}084=-0{,}001748$;

$\overline{Q}\left(=\frac{2Q}{qe^4f}+\frac{Q'}{e^4f}\right)=-\frac{0{,}0084}{3}=-0{,}0028$;

$\overline{PB1}\left(=\frac{4}{p-\beta}+\frac{3}{2}-\frac{\beta}{2}-\frac{1-\beta\beta}{p}\times\frac{PB}{e^4f}\right)=-0{,}048B$;

$\overline{PB2}\left(=\frac{4}{p+\beta}+\frac{3}{2}+\frac{\beta}{2}-\frac{1-\beta\beta}{p}\times\frac{PB}{e^4f}\right)=-0{,}0285B$;

$\overline{PC2}\left(=\frac{4}{p+\gamma}+\frac{3}{2}+\frac{\gamma}{2}-\frac{1-\gamma\gamma}{p}\times\frac{PC}{e^4f}\right)=-0{,}0404C$;

$\overline{PD1}\left(=\frac{4}{p-\delta}+\frac{3}{2}-\frac{\delta}{2}-\frac{1-\delta\delta}{p}\times\frac{PD}{e^4f}\right)=-0{,}0414D$;

$\overline{PD2}\left(=\frac{4}{p+\delta}+\frac{3}{2}+\frac{\delta}{2}-\frac{1-\delta\delta}{p}\times\frac{PD}{e^4f}\right)=-0{,}0273D$;

$\overline{QB1}\left(=\frac{4Q}{q-\beta}+\frac{3Q'}{2}-\frac{\beta Q}{2}-\overline{1-\beta\beta}\times\frac{Q}{q}\times\frac{B}{e^4f}\right)=\frac{PB}{2e^4f}=-$ $0{,}0042B$ (becauſe $Q=0$, and $Q'=\frac{1}{3}P$); and in the very ſame manner, $\overline{QB2}=-0{,}0042B$; $\overline{QC1}=-0{,}0042C$; $\overline{QC2}=-0{,}0042C$; $\overline{QD1}=-0{,}0042D$, $\overline{QD2}=-0{,}0042D$, &c. &c. *.

Whence

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* In theſe calculations, all terms whoſe diviſors are found equal to nothing, will themſelves be nothing, and not infinite, as might at firſt be imagined: thus the term $\frac{\text{coſ.}\overline{p-\gamma}\,z}{p-\gamma}$, by having its diviſor $p-\gamma=0$, intirely vaniſhes. The reaſon whereof will appear evident, if it be conſidered, that this term, $\frac{\text{coſ}\,\overline{p-\gamma}.z}{p-\gamma}$, ariſes by taking rhe fluent of $-\dot{z}\times\text{ſin.}\,\overline{p-\gamma}.z$, or $-\dot{z}\times 0$: which is, manifeſtly, equal to nothing. But an objection may, perhaps, be brought from hence, againſt the truth and univerſality of the *fluxionary calculus*; ſeeing thereby an expreſſion is here derived, which, though actually equal to nothing, appears nevertheleſs under the form of an infinite quantity. To clear up this point, it muſt be obſerved, that $\frac{\text{coſ.}\overline{p-\gamma}\,z}{p-\gamma}$ is not the *complete* fluent of $-\dot{z}\times\text{ſin.}\,\overline{p-\gamma}.z$, but the variable part of it only; the *corrected* fluent being $-\frac{1}{p-\gamma}+\frac{\text{coſ.}\overline{p-\gamma}.z}{p-\gamma}$, or $\frac{1}{p-\gamma}\times\overline{-1+\text{coſ.}\overline{p-\gamma}.z}$. But, whatever value $p-\gamma$ is ſuppoſed to have, the coſine of $\overline{p-\gamma}.z$ will, it is known, be expreſſed

Whence it is evident, that

$B'\,(=\overline{QB1}+\overline{QB2}+\overline{PD1}\ \&c.)=-0{,}0084B-0{,}0414D,$

$C'\,(=\overline{P}+\overline{QC1}+\overline{QC2}\ \&c.)=-0{,}01748-0{,}0084C,$

$D'\,(=\overline{PB1}+\overline{QD1}+\overline{QD2}\ \&c.)=-0{,}048B-0{,}0084D,$

$E'\,(=\overline{PB2}\ \&c.)=-0{,}0285B,$

$F'\,(=\overline{PC2}\ \&c.)=-0{,}0404C,$

$G'\,(=\overline{PD2}\ \&c.)=-0{,}0273D,$ &c.

But B' being $=-\overline{1-\beta\beta}.B$, $C'=-\overline{1-\gamma\gamma}.C$, $D'=-\overline{1-\delta\delta}.D$, &c. the ſecond and third of theſe equations, will by ſubſtitution, be changed to $\overline{1-\gamma\gamma}.C=0{,}01748+0{,}0084C$, and $\overline{1-\delta\delta}.\ D=0{,}048\,B+0{,}0084D$; whence C is given $=\frac{0{,}01748}{1-\gamma\gamma-0{,}0084}=\frac{0{,}01748}{1-\overline{1{,}8504}|^2-0{,}0084}=-0{,}007186$; and $D=\frac{0{,}048\,B}{1-\overline{0{,}8504}|^2-0{,}0084}=0{,}178\,B$: which values being ſubſtituted in the other equations, they will become $B'=-0{,}0158B$, $E'=-0{,}0285B$, $F'=0{,}0003$, $G'=-0{,}005\,B$; whence $E\,(=-\frac{E'}{1-\overline{p+\beta}|^2})=-0{,}004B$, $F\,(=-\frac{F'}{1-4pp})=0{,}000023$, $G\,(=-\frac{G'}{1-\overline{2p-\beta}|^2})=-0{,}0008\,B$; and conſequently $1-w=(e\times\overline{1-B\,\text{coſ.}\,\beta z-C\,\text{coſ.}\,\gamma z-D\,\text{coſ.}\,\delta z\ \&c.})=e$ into $1-B\,\text{coſ.}\,\beta z+0{,}007186\,\text{coſ.}\,pz-0{,}178\,B\,\text{coſ.}\,\overline{p-\beta}.z+0{,}004B\,\text{coſ.}\,\overline{p+\beta}.z-0{,}000023\,\text{coſ.}\,2pz+0{,}0008B\,\text{coſ.}\,\overline{2p-\beta}.z$ &c. expreſſing the reciprocal of the moon's diſtance from the earth.

As to the quantity B, which enters into the greater part of the terms of the ſeries here found, it cannot, as has been al-

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preſſed by $1-\frac{\overline{p-q}|^2z^2}{1.\,2}+\frac{\overline{p-\gamma.}|^4z^4}{1.\,2.\,3.\,4}$ &c. So that, the fluent in queſtion will be truly repreſented by $\frac{1}{p-\gamma}\times-\frac{\overline{p-\gamma.}|^2z^2}{1.\,2}+\frac{\overline{p-\gamma.}|^4z^4}{1.\,2.\,3.\,4}$ &c. or its equal $-\frac{\overline{p-\gamma}.z^2}{1.\,2}+\frac{\overline{p-\gamma}|^3z^4}{1.\,2.\,3.\,4}$ &c. which entirely vaniſhes, as it ought to do, when $p-\gamma=0$.

ready intimated, be otherwiſe determined than from the obſervations of *Aſtronomers:* nor will the equation above expreſſing the relation of B' and B, afford us the leaſt help therein: For, by ſubſtituting $-0{,}0158\,B$ inſtead of its equal B', that equation will become $-0{,}0158\,B=-\overline{1-\beta\beta}.B$, or $1-\beta\beta=0{,}0158$; where, B intirely vaniſhing, nothing in relation to it can, therefore, be determined. We have here, indeed, an equation for finding the value of β; which from thence is given $=\sqrt{1-0{,}0158}=0{,}99206$; by means whereof the motion of the *apogee* will be known: for it will appear, *by cor.* 1 *and* 3 *to prob.* VI.) that $1-w=e\times\overline{1-B\,\text{coſ.}\,\beta z}$ is the equation correſponding to a moveable ellipſe, turning about the focus, or center of force, with an angular celerity which is to that of the body in the ellipſe, every-where in the conſtant proportion of $1-\beta$ to β: whence it follows, that the mean motion of the *apogee*, ought to be in proportion to the mean motion of the moon, as 0,00794 to *unity:* which differs from the *real* proportion (of 0,008455 to 1) *as given from obſervation*, by about $\frac{1}{18}$ part of the whole value: nor ought this to ſeem ſtrange, as a number of (ſmall) terms yet remain to be introduced into the value of $1-w$, by the corrections pointed out on *p.* 156 and 157: beſides which, the difference ariſing in the co-efficients of the terms already found, by ſubſtituting this, *new*, value for β, will amount to ſomething conſiderable, from whence, alone, near half the error would be taken away. But, to avoid the trouble of repeating the ſame operation, again and again, with the new values of β, thus found, I ſhall here, at once, take β equal to the true value (0,991545) as given from *obſervation*; but

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If the eccentricity B, be ſuppoſed to vaniſh, $1-w$ *will then become* $=e\times\overline{1+0{,}007186\,\text{coſ.}\,pz-0{,}000023\,\text{coſ.}\,2pz\ \&c.}$ *which, when* $pz=0$, *or the moon is the ſyzigy, will be* $=e\times\overline{1+0{,}007186-0{,}000023}=1{,}007163\times e$; *but when* $\frac{1}{2}pz=90$ *degrees, or the moon is in the quadrature, it will then be* $=e\times\overline{1-0{,}007186-0{,}000023}=0{,}992791\times e$. Therefore, when the orbit is without excentricity, the diſtance *of the moon from the earth, in the ſyzigy is in proportion to the diſtance in the quadrature, as* $\frac{1}{1{,}007163\times e}$ *to* $\frac{1}{0{,}992791\times e}$, *or as* 0,992791 *to* 1,007163, *that is, as* 69 *to* 70, *very near; the ſame as is found by Sir Iſaac Newton in his Princip. B.* III. *prop.* 28.

ſhall, at the ſame time, put down the ſeveral terms ariſing in the equation for the value of β; by means whereof it will appear, in caſe both ſides are found to be equal, that the value of the root β has been rightly aſſumed; and that the motion of the moon's *apogee* (which has been the ſubject of ſo much ſpeculation and controverſy) is intirely conſiſtent with the general laws of gravitation.

Now, of the quantities above determined, $\overline{PB1}$ and $\overline{PD1}$ are thoſe that are the moſt affected by altering the value of β: theſe being, therefore, computed a-new (making β equal to 0,991545, inſtead of unity) the former will here come out $= -0{,}04747\,B$, and the latter $= -0{,}04168\,D$: from whence, by proceeding as before, we have $D = 0{,}1869\,B$, and $1-\beta\beta\left(=-\frac{B'}{B}\right)=0{,}01619$. As to the values of E, F, G, &c. they are ſo ſmall, in themſelves, and ſo little affected by β, that to compute them a-new, would be quite unneceſſary; the difference not producing an error of a ſingle *ſecond*, in the place of the moon.

To apply, now, the obſervations laid down at *page* 156, in order to obtain from thence a farther correction of the value $\left(e \times \left\{ \begin{array}{l} 1 - B\,\text{coſ.}\,\beta z + 0{,}007186\,\text{coſ.}\,pz - 0{,}1869\,B\,\text{coſ.}\,\overline{p-\beta}.z \\ + 0{,}004\,B\,\text{coſ.}\,\overline{p+\beta}.z - 0{,}000023\,\text{coſ.}\,2pz + 0{,}0008\,B\,\text{coſ.}\,\overline{2p-\beta}.z \end{array}\right.\right)$ above found, the ſeveral powers of the ſeries $B\,\text{coſ.}\,\beta z - 0{,}007186\,\text{coſ.}\,pz + 0{,}1869\,B\,\text{coſ.}\,\overline{p-\beta}.z$ &c. (before omitted) muſt be now taken, or ſuch terms thereof, at leaſt, as are of conſequence enough to merit regard. Thus, in the ſquare, or ſecond power, the terms which appear conſiderable enough to merit an examination, at leaſt, will be thoſe ariſing from the ſquares and the double rectangles of the three firſt terms of the root, which are vaſtly larger than the others. Theſe will be $\frac{1}{2}B^2 + \frac{1}{2}B^2\,\text{coſ.}\,2\beta z - {,}007186\,B\,\text{coſ.}\,\overline{p-\beta}.z - {,}007186\,\text{coſ.}\,\overline{p+\beta}.z + 0{,}1869\,B^2\,\text{coſ.}\,\overline{p-2\beta}.z + 0{,}1869\,B^2\,\text{coſ.}\,pz + {,}000025 + {,}000025\,\text{coſine}\,2pz - {,}00135\,B\,\text{coſine}\,\beta z - {,}00135\,B\,\text{coſ.}\,\overline{2p-\beta}.z + {,}0175\,B^2 + {,}0175\,B^2\,\text{coſ.}\,\overline{2p-2\beta}.z$. But, in the value of $\overline{1-w}|^{-3}$, theſe terms are affected with the common multiplicator $\frac{6}{e^3}$ (*vide p.* 154); and, in the value of

$\overline{1-w}|^{-4}$, by the common multiplicator $\frac{10}{e^4}$: ſo that, in order to find the effect of the firſt of them ($\frac{1}{2}B^2$) from the *formula* at *p.* 156, we muſt compare $\frac{1}{2}B^2\times\frac{6}{e^3}$ and $\frac{1}{2}B^2\times\frac{10}{e^4}$, with $\frac{M}{e^3}\times$ coſ. αz, and $\frac{N}{e^4}\times$ coſ. αz, reſpectively; whence $M=3B^2$, $N=5B^2$, and $\alpha=0$: and conſequently,

$$\frac{P}{e^4 f}\times\overline{\frac{N}{p-\alpha}+\frac{MP'}{2P}-\frac{\alpha M}{6}}\times\frac{\text{coſ.}\,\overline{p-\alpha}.z}{1-\overline{p-\alpha}|^{2}}+$$
$$\frac{P}{e^4 f}\times\overline{\frac{N}{p+\alpha}+\frac{MP'}{2P}+\frac{\alpha M}{6}}\times\frac{\text{coſ.}\,\overline{p+\alpha}.z}{1-\overline{p+\alpha}|^{2}}\ \text{\&c.}={,}029\,B^2\ \text{coſ.}\ pz.$$

In the ſame manner, with reſpect to the ſecond term, $\frac{1}{2}B^2$ coſ. $2\beta z$, we have $M=3B^2$, $N=5B^2$ (as before), and $\alpha=2\beta$; from whence the correction, anſwering thereto (excluſive of the general factor e) will come out, $0{,}3178\,B^2$ coſ. $\overline{p-2\beta}.z$ + $,0023\,B^2$ coſ. $\overline{p+2\beta}.z+{,}0028$ coſ. $2\beta z$.

Again, in relation to $-{,}007186\,B$ coſ. $\overline{p-\beta}.z$, we have $M=-{,}431\,B$, $N=-{,}0718\,B$, and $\alpha=p-\beta$; from which will be found $,000735\,B\times\frac{\text{coſ.}\,\beta z}{1-\beta\beta}-{,}000073\,B$ coſ. $\overline{2p-\beta}.z$ $+{,}000464\,B$coſ.$\overline{p-\beta}.z$; where the firſt term is of that ſpecies on which the motion of the *apogee* depends, and where the *ſecond* is too ſmall to be farther regarded.

In like ſort, from $-{,}007186\,B$ coſ. $\overline{p+\beta}.z$, we ſhall get $-{,}00060\,B\times\frac{\text{coſ.}\,\beta z}{1-\beta\beta}$, beſides two other terms; which, on account of their extreme ſmallneſs, may be intirely neglected.

By proceeding on, in this manner, two, or three ſmall terms, more, (producing equations of a few ſeconds, each) will be found; which being joined to thoſe above, we ſhall from thence have, $+{,}03\,B^2$ coſ. $pz+{,}00055\,B$ coſ. $\overline{p-\beta}.z+$ $0{,}3028\,B^2$coſ.$\overline{p-2\beta}.z+{,}0072\,B^2$coſ. $2\beta z+{,}0070\,B^2$ coſ.$\overline{2p-2\beta}.z$ $+{,}00016\,B\times\frac{\text{coſ.}\,\beta z}{1-\beta\beta}$; which, drawn into the general multiplicator e, will give the whole increaſe of $1-w$, by taking in the ſquare of the ſeries aforeſaid. As to the increaſe ariſing from the cube thereof, few of the terms will be found conſiderable enough to deſerve notice, the only ones of any conſequence to be collected from thence, being $-\overline{,00004B+0,7B^3}\times\text{coſ.}\overline{p-\beta.z}$ $-0,11B^3\times\text{coſine }\overline{p-3.\beta.z}-,08\,B^3\times\text{coſ. }\overline{2p-3\beta.z}$ $-,055B^3\times\frac{\text{coſ.}\beta z}{1-\beta\beta}$: But, from the *biquadrate*, we ſhall only have one term ($1,1B^4\,\text{coſ.}\overline{p-2\beta.z}$) that can produce an equation of more than a *ſecond*, or two; though the effect of this one term will, alone, amount to near a *minute*; which is owing to the ſmallneſs of the exponent $p-2\beta$, conſonant to what has been before inculcated (at *page* 157). And it may be worth while to take farther notice here, that all the terms hitherto determined, except two or three, belong to one, or the other of the two *kinds* there ſpecified.

If, now, the ſeveral quantities above brought out, be collected into one ſum, we ſhall have $0,03B^2\,\text{coſ.}pz+\overline{,00051B-0,7B^3}$ $\times\text{ coſine }\overline{p-\beta.z}+\overline{0,3028B^2+1,1B^4}\times\text{ coſine }\overline{p-2\beta.z}$ $+,0072B^2\,\text{coſ.}2\beta z+,0070B^2\,\text{coſ.}\overline{2p-2\beta.z}-0,11B^3\,\text{coſ.}\overline{p-3\beta.z}$ $-,08\,B^3\,\text{coſ.}\overline{2p-3\beta.z}+\overline{,00016B-,055B^3}\times\frac{\text{coſ.}\beta z}{1-\beta\beta}$; of all which terms, $\overline{0,3028B^2+1,1B^4}\times\text{coſ.}\overline{p-2\beta.z}$ is by far the greateſt, and ſo conſiderable as to produce other terms, or corrections, of conſequence enough to be regarded here. The new terms ariſing from the entering of this term into the firſt power of the ſeries, for the value of $1-w$, will appear to be $-,0032B^2\,\text{coſ.}2\beta z-,0047B^2\,\text{coſ.}\overline{2p-2\beta.z}+,0026B^2\,\text{coſ.}\overline{p-2\beta.z}$ (which are found as above, by making $M=3\times\overline{-0,3028B^2}$, $N=4\times\overline{-0,3028B^2}$, $\alpha=p-2\beta$, and neglecting $1,1\,B^4$): And by the entering thereof into the ſquare of that ſeries, from the double rectangle under it and the firſt term, there will be had $+,088\,B^3\,\text{coſ. }\overline{2p-3\beta.z}$ $-,019B^3\,\text{coſ.}\overline{p-3\beta.z}+,03B^3\times\frac{\text{coſ. }\beta z}{1-\beta\beta}$: which, together with the quantities before found, being joined to the former value of $1-w$, we have at length

$$1-w=e\times\begin{cases}1-B\,\text{coſ}.\beta z+\overline{,007186+,03B^2}\times\text{coſ}.pz-\overline{0,1864B+0,7B^3}\times\text{coſ}.\overline{p-\beta.z}\\ +,004B\,\text{coſ}.\overline{p+\beta.z}+\overline{0,3054B^2+1,1B^4}\times\text{coſ}.\overline{p-2\beta.z}-,000023\,\text{coſ}.2pz\\ +,0008B\,\text{coſ}.\,\overline{2p-\beta.z}+,004B^2\,\text{coſ}.\,2\beta z+,0023B^2\,\text{coſ}.\,\overline{2p-2\beta.z}\\ \qquad [+0,13B^3\,\text{coſ}\,\overline{p-3\beta.z}\end{cases}$$

In which equation the terms of the ſpecies, coſ. βz, are not put down; becauſe the original term $-B$ coſ. βz, which is equivalent to them, is ſtill retained.

Thoſe terms, however, though of no uſe in this equation, are not to be intirely diſregarded; ſince on them the motion of the *apogee* depends. By collecting them, and the former value of $-B$, into one ſum, we get the equation $-B=-\frac{,01619B}{1-\beta\beta}+\frac{,00016B-,055B^3}{1-\beta\beta}+\frac{,03B^3}{1-\beta\beta}=-\frac{,01603B+,025B^3}{1-\beta\beta}$: from whence, when B is aſſigned, the value of β will be found, but *ſtill* ſomething ſhort of the true value, as given from *obſervation*.

But it ought to be now remembered, that all the equations hitherto brought out, are derived on the hypotheſis, that the *true motion* of the ſun is to *that* of the moon, in a conſtant proportion: which is near the truth, only; ſince the diſtance of the ſun from the moon, whereon the perturbating forces depend, is ſometimes greater, or leſs by two degrees, than according to that hypotheſis: which difference muſt, of conſequence, render other corrections neceſſary. How *theſe* may be introduced I ſhall here inſtance, by firſt conſidering the motion of the earth about the ſun, as uniformly performed in a circular orbit; leaving, to be determined by a future operation, the other equations ariſing from the eccentricity and parallax, which will be obtained in the ſame manner.

It is found at *page* 159, that the moon's *mean motion*, anſwering to the *true motion* z, is rightly repreſented by $z+\frac{(B)\times\text{ſin}.\beta z}{b\beta}+\frac{(C)\times\text{ſin}.\gamma z}{b\gamma}+\frac{(D)\times\text{ſin}.\delta z}{b\delta}$ &c; from whence the *mean*, or *true motion* of the ſun (for, in a circular orbit, they are the ſame) will be had $=m\times\overline{z+\frac{(B)\times\text{ſin}.\beta z}{b\beta}+\frac{(C)\times\text{ſin}\,\gamma z}{b\gamma}}$ &c. $=mz+mz'$, ſuppoſing $z'=\frac{(B)\times\text{ſin}.\beta z}{b\beta}+\frac{(C)\times\text{ſin}.\gamma z}{b\gamma}$ &c. Hence the true diſtance of the moon from the ſun, will be $z - mz - mz'$, and the double thereof equal to $\overline{1-m}\times 2z - 2mz'$, or to $pz - 2mz'$: of which the ſine is $= \text{ſin.}\,pz \times \text{coſ.}\,2mz' - \text{coſ.}\,pz \times \text{ſin.}\,2mz'$; but the arch $2mz'$ being exceeding ſmall, the ſine thereof will be nearly equal to the arch itſelf, and the coſine very nearly equal to the radius, or *unity*; and conſequently the ſine of the double diſtance of the moon from the ſun $= \text{ſin.}\,pz - \text{coſ.}\,pz \times 2mz' = \text{ſin.}\,pz - \text{coſ.}\,pz \times \frac{2m(\text{B})\times\text{ſin.}\,\beta z}{b\beta}$ &c. $= \text{ſin.}\,pz + \frac{m(\text{B})}{b\beta} \times \overline{\text{ſin.}\,\overline{p-\beta}.z - \text{ſin.}\,\overline{p+\beta}.z} + \frac{m\,(\text{C})}{b\gamma} \times \overline{\text{ſin.}\,\overline{p-\gamma}.z - \text{ſin.}\,\overline{p+\gamma}.z}$ &c.

In like manner, the co-ſine thereof will be had ($= \text{coſ.}\,pz \times \text{coſ.}\,2mz' + \text{ſin.}\,pz \times \text{ſin.}\,2mz' = \text{coſ.}\,pz + \text{ſin.}\,pz \times 2mz'$ nearly) $= \text{coſ.}\,pz + \frac{m(\text{B})}{b\beta} \times \overline{\text{coſ.}\,\overline{p-\beta}.z - \text{coſ.}\,\overline{p+\beta}.z} + \frac{m\,(\text{C})}{b\gamma} \times \overline{\text{coſ.}\,\overline{p-\gamma}.z - \text{coſ.}\,\overline{p+\gamma}.z}$ &c.

Which two quantities being multiplied by P, it will appear, that the new terms ariſing in the two expreſſions of the forces (beſides thoſe, $\text{P ſin.}\,pz$ and $\text{P coſ.}\,pz$. in the former hypotheſis) are $\frac{m\text{P}\,(\text{B})}{b\beta} \times \overline{\text{ſin.}\,\overline{p-\beta}.z - \text{ſin.}\,\overline{p+\beta}.z} + \frac{m\text{P}\,(\text{C})}{b\gamma} \times \overline{\text{ſin}\,\overline{p-\gamma}.z - \overline{p+\gamma}.z}$ &c. and $\frac{m\text{P}\,(\text{B})}{b\beta} \times \overline{\text{coſ.}\,\overline{p-\beta}.z - \text{coſ.}\,\overline{p+\beta}.z} + \frac{m\text{P}(\text{C})}{b\gamma} \times \overline{\text{coſ.}\,\overline{p-\gamma}.z - \overline{p+\gamma}.z}$ &c.

Now it will appear, by a compariſon with the *formula* on *p.* 157, that the effect of the two firſt, correſponding terms here exhibited, making $\pi = p - \beta$, will be $\frac{m\text{P}\,(\text{B})}{b\beta e^4 f} \times \overline{\frac{2}{\pi} + 1} \times \frac{\text{coſ.}\,\pi z}{1-\pi\pi} + \overline{\frac{4}{\pi-\beta} + \frac{3}{2} - \frac{\beta}{2} - \frac{1-3\beta}{\pi}} \times \frac{m\text{PB}\,(\text{B})}{b\beta e^4 f} \times \frac{\text{coſ.}\,\overline{\pi-\beta}.z}{1-\overline{\pi-\beta}|^2} + \overline{\frac{4}{\pi+\beta} + \frac{3}{2} + \frac{\beta}{2} - \frac{1-3\beta}{\pi}} \times \frac{m\text{PB}\,(\text{B})}{b\beta e^4 f} \times \frac{\text{coſ.}\,\overline{\pi+\beta}.z}{1-\overline{\pi+\beta}|^2}$ &c. which, becauſe $m = {,}0748$, $\frac{\text{P}}{e^4 f} = -{,}0084$, and $b = 1$, will be equal to

$-\frac{,00063(B)}{\beta}$ into $\overline{\frac{2}{\pi}+1}\times\frac{\text{coſ. }\pi z}{1-\pi\pi}+\overline{\frac{4}{\pi-\beta}+\frac{3}{2}-\frac{\beta}{2}-\frac{1-\beta\beta}{\pi}}$ $\times\frac{B\text{coſ. }\overline{\pi-\beta}.z}{1-\overline{\pi-\beta}|^2}+\overline{\frac{4}{\pi+\beta}+\frac{3}{2}+\frac{\beta}{2}-\frac{1-\beta\beta}{\pi}}\times\frac{B\text{coſ. }\overline{\pi+\beta}.z}{1-\overline{\pi+\beta}|^2}+$ $\overline{\frac{4}{\pi-\gamma}+\frac{3}{2}-\frac{\gamma}{2}-\frac{1-\gamma\gamma}{\pi}}\times\frac{C\text{coſ. }\overline{\pi-\gamma}.z}{1-\overline{\pi-\gamma}|^2}+$ &c. $=-,0081\ (B)$ $\times$ coſine $\overline{p-\beta}.z+,0187B\ (B)\times\text{coſ. }\overline{p-2\beta}.z+,0011\ B(B)$ $\times\text{coſ. }pz$ &c. where all the terms, after the two firſt, may, on account of their extreme ſmallneſs, be intirely neglected.

In the very ſame manner, expounding π by $p+\beta$, $p-\gamma$ $(=0)$, $p+\gamma$ $(=2p)$, $p-\delta$ $(=\beta)$, and $p+\delta$ $(=2p-\beta,)$ ſucceſſively, and ſubſtituting $+\frac{,00063(B)}{\beta}$, $-\frac{,00063(C)}{\gamma}$, $+\frac{,00063(C)}{\gamma}$, $-\frac{,00063(D)}{\delta}$, and $+\frac{,00063(D)}{\delta}$, reſpectively, in the room of $-\frac{,00063(B)}{\beta}$; there will, *in the firſt caſe*, be found the term $-,00015\ (B)\times\text{coſ. }\overline{p+\beta}.z$; being the only one producing an equation of more than a ſingle ſecond; *in the ſecond and third caſes*, no term at all, worth notice, will be found: but, *in the fourth* (where $\pi=\beta$) two pretty conſiderable ones $,00222\ (D)\times\frac{\text{coſ. }\beta z}{1-\beta\beta}$, and $-0,0233\ D(D)\times\text{coſ. }\overline{p-2\beta}.z$, do ariſe; whereof the former is of that ſpecies on which the motion of the *apogee* depends, and gives, an increaſe of that motion, of about $\frac{1}{20}$ part. By purſuing the ſame method, the effect of the ſecond order of terms, $\frac{(BB)\times\text{ſin. }2\beta z}{h.2\beta}$, $\frac{(BC)\times\text{ſin. }\overline{\gamma-\beta}.z}{h.\overline{\gamma-\beta}}$ &c. may alſo be computed; but here it will be ſufficient to make uſe of the firſt term of the general *formula* alone: from whence are obtained the quantities $+,0045\ (BB)\times\text{coſ. }\overline{p-2\beta}.z$, and $-,0059\ (DD)$ $\times\text{coſ. }\overline{p-2\beta}.z$; which are both of that kind, ſpecified at *p.* 157, rendered conſiderable by the ſmallneſs of their exponents, and are the only ones *here*, that merit regard. Theſe being, therefore, joined to thoſe found above, the whole correction ſought will be, $-0,0081(B)\times\text{coſ. }\overline{p-\beta}.z-0,00015\ (B)\times\text{coſ. }\overline{p+\beta}.z$

$+\overline{0{,}0187B(B)-0{,}233D(D)+0{,}0045(BB)-0{,}0059(DD)}\times\text{coſ.}\,p-2\beta.z$. But (B) being $=2B$, $(D)=2D$ $(BB)=\frac{3}{2}B^2$, and $(DD)=\frac{3}{2}D^2$, nearly, *vid. p.* 159 ($\bar{B}$, $\bar{C}$, $\bar{D}$, &c. being neglected, as too inconſiderable to be regarded here) our expreſſion may therefore be changed to $-0{,}0162\,B\,\text{coſ.}\,\overline{p-\beta}.z-0{,}0003\,B\,\text{coſ.}\,\overline{p+\beta}.z+\overline{0{,}0441\,B^2-0{,}0554D^2}\times\text{coſ.}\,\overline{p-2\beta}.z$ which being added to the value of $1-w$, before found, we thence have

$$1-w=e\times\left\{\begin{array}{l}1-B\,\text{coſ.}\,\beta z+\overline{0{,}007186+0{,}03B^2}\times\text{coſ.}\,pz-\overline{0{,}2026B+0{,}7B^3}\times\text{coſ.}\,\overline{p-\beta}.z\\+0{,}0037B\,\text{coſ.}\,\overline{p+\beta}.z+\overline{0{,}03495B^2-0{,}0554D^2+1{,}1B^4}\times\text{coſ.}\,\overline{p-2\beta}.z\\-{,}000023\,\text{coſ.}\,2pz+{,}0008\,B\,\text{coſ.}\,\overline{2p-\beta}.z+{,}004B^2\,\text{coſ.}\,2\beta z+\\{,}0023\,B^2\,\text{coſ.}\,\overline{2p-2\beta}.z+0{,}13B^3\,\text{coſ.}\,\overline{p-3\beta}.z\end{array}\right.$$

And, by writing the coefficient of the term $-\frac{0{,}00222(D)\times\text{coſ.}\,\beta z}{1-\beta\beta}$, in the equation for the motion of the *apogee*, it will here become $-B=-\frac{0{,}01603B+0{,}025B^3}{1-\beta\beta}-\frac{0{,}00222(D)}{1-\beta\beta}=-\frac{0{,}01693B+0{,}028B^3}{1-\beta\beta}$; becauſe $(D)=2D=2\times\overline{0{,}2026B+0{,}7B^3}$: whence $1-\beta\beta=0{,}01693+00028B^2=0{,}017015$ (ſuppoſing $B=0{,}05505$); and conſequently $1-\beta=0{,}00854$; which value is, now, about $\frac{1}{100}$ part greater than the true value, given *from obſervation*, and is near enough to ſhew, that the force of the ſun is ſufficient to produce all the motion of the moon's *apogee*, without ſuppoſing a change in the general law of gravitation, from the inverſe ratio of the ſquares of the diſtances. Several very ſmall terms belonging to this equation, have been omitted, beſides thoſe ariſing from the ſun's *excentricity*, and the *inclination* of the *lunar orbit*; which together, may very well be ſuppoſed, ſufficient to cauſe a difference equal to that abovementioned.

If, now, the value of, B expreſſing the *mean excentricity*, in *Sir Iſaac Newton's* lunar theory, be expounded by 0,05505, *according to that author*, the general equation of the orbit, found above, will here become

$$1-w=e\times\left\{\begin{array}{l}1-0{,}05505\,\text{coſ.}\,\beta z+0{,}007276\,\text{coſ.}\,pz-0{,}01127\,\text{coſ.}\,\overline{p-\beta}.z\\+0{,}000204\,\text{coſ.}\,\overline{p+\beta}.z+0{,}0010623\,\text{coſ.}\,\overline{p-2\beta}.z-0{,}000023\,\text{coſ.}\,2pz\\+0{,}000044\,\text{coſ.}\,\overline{2p-\beta}.z+0{,}000012\,\text{coſ.}\,2\beta z+0{,}000007\,\text{coſ.}\,\overline{2p-2\beta}.z\\+0{,}000021\,\text{coſ.}\,\overline{p-3\beta}.z\end{array}\right.$$

But this value muſt now be corrected by the difference ariſing from our having, in all the preceding calculations, taken the diviſor $hh=1$, inſtead of the true value $\overline{1+\frac{1}{2}\times\text{BB}+\text{CC}+\text{DD\&c.}}|^{2}$ (*vid. p.* 159.) which value, becauſe $\text{B}=0{,}05505$, $\text{C}=-0{,}007276$ &c. is given $=1{,}0097$. From whence and the equations, on *p.* 153 and 163, it appears, that all the terms above exhibited, in whoſe exponents the quantity p is, *ſingly*, concerned, ought to be diminiſhed in the ratio of 1 to 1,0097; and that all *thoſe*, where $2p$ is in like manner concerned, ought to be diminiſhed, in the duplicate of that ratio: by which means our equation is, at length, reduced to

$$1-w=e\times\left\{\begin{array}{l}1-0{,}05505\,\text{coſ.}\,\beta z+0{,}007206\,\text{coſ.}\,pz-0{,}01116\,\text{coſ.}\,\overline{p-\beta}.z\\+0{,}000202\,\text{coſ.}\,\overline{p+\beta}.z+0{,}001052\,\text{coſ.}\,\overline{p-2\beta}.z-0{,}000022\,\text{coſ.}\,2pz\\+0{,}000043\,\text{coſ.}\,\overline{2p-\beta}.z+0{,}000012\,\text{coſ.}\,2\beta z+0{,}000007\,\text{coſ.}\,\overline{2p-2\beta}.z\\+0{,}000021\,\text{coſ.}\,\overline{p-3\beta}.z.\end{array}\right.$$

From whence all the great *equations* of the moon's motion, and all the ſmaller ones, except thoſe depending on the ſun's *excentricity* &c. are obtained, within leſs than half a *minute* of the truth; ſuppoſing the mean excentricity (B) to be here truly aſſigned. If it ſhould be found neceſſary to augment, or diminiſh the value thereof, then the term $-0{,}01116\,\text{coſ.}\,\overline{p-\beta}.z$ (producing the *equation*, called the *evection*) muſt be alſo augmented or diminiſhed, in the ſame ratio; and the term $+0{,}001052\,\text{coſ.}\,\overline{p-2\beta}.z$ (which is the next conſiderable of thoſe wherein β enters) muſt be augmented or diminiſhed in the duplicate of that ratio. As to the reſt of the terms, they are ſo ſmall, that a little alteration in the value of B will produce no difference in them worth notice.

In the ſame manner, the inequalities cauſed in the moon's motion by the ſun's excentricity, may be computed. For the mean motion of the moon being given, very nearly, by the preceding calculations, the mean motion of the ſun, being in proportion thereto as m to 1, will be alſo known; from whence, and the *excentricity*, the ſun's *true anomaly*, and diſtance from the earth will be had, in a ſeries of ſines and co-ſines of the multiples of the arch z; whoſe exponents and coefficients are all intirely known: by means of which, the ſun's *true diſtance* from the moon being (nearly) obtained, the *new* terms ariſing from the eccentricity, in the general expreſſions for the perturbating forces, will be had, in conſequence thereof; and, laſtly, the effects themſelves, produced thereby in the general equation of the orbit.

When the equation of the orbit, or the value of $1-w$ (the reciprocal of the moon's diſtance from the earth) is thus determined to a ſufficient degree of exactneſs (by repeating the operation if neceſſary), the difference between the *true*, and *mean motions* of the moon, muſt, from thence, be found, in terms of the latter: this may be done by firſt finding the *mean motion* in terms of the *true*. (as is ſhewn at *p.* 159) and then reverting the ſeries; or, otherwiſe, without finding the *mean motion* at all, by the reſolution of the fluxionary equation $\dot{t} = \frac{g}{2c} \times \frac{\dot{z}}{\Sigma^{\frac{1}{2}} \times \overline{1-u}|}$; according to the method made uſe of of in a former *paper*, in this collection. The proceſs (which is more laborious than difficult) I ſhall not inſert here, as my deſign, in this *miſcellaneous work*, is not to exhibit every operation neceſſary to the forming of a *complete* theory of the moon's motion; which, for its importance, and the great variety and intricacy of calculations ariſing therein, may very well merit to be the ſubject of a *volume*, by itſelf.

It may ſuffice, in this place, to have pointed out (by a method not very perplexed) how the different *inequalities* of that motion may be determined, and the moon's true place, *according to gravity*, aſcertained. At another time, if health permits, I may, not only give, at large, the application of the *equations* and *precepts* here delivered, but alſo a new ſet of *lunar tables*, deduced therefrom; which (though a work of much labour) I ſhall the more chearfully undertake, as *Dr. Bradley* has very obligingly offered to aſſiſt me with any obſervations, that may be wanting in order to the compleating of the deſign, and eſtabliſhing the *theory* on a proper *baſis*. And I have ſome reaſon to hope, that, when that part of the *data*, which can be only known from obſervations, is truly ſettled, the place of the moon may be *always* obtained, within about a *minute* of the truth, even, without the uſe of a multitude of tables, by means of proper contrivances, reſpecting the ſmaller equations, or the leſs conſiderable terms of the general ſeries.

I ſhall conclude what I have to ſay on this ſubject, at preſent, with obviating a difficulty in the application of the equations abovementioned, when the effect in the moon's motion, depending on the ſun's *eccentricity* and *parallax* *, are computed thereby. In this caſe, a new ſpecies of (very ſmall) terms, affected with the co-ſine of the arch z, will be found to enter into the general equation (*on page* 151) whereof the effect cannot be determined in the ſame manner with that of the other terms, affected with the co-ſines of the multiples of that arch: For if, according to the method of proceeding there laid down, we aſſume a term (as $\overline{1-\pi\pi}.\Pi\ \text{coſ.}\ \pi z$) in the upper line, $\overline{1-\beta\beta}.\text{B coſ.}\ \beta z+\overline{1-\gamma\gamma}.\text{C coſ.}\ \gamma z+\overline{1-\delta\delta}.\text{D coſ.}\ \delta z$ &c. in order to compare it with a term, g coſ. z, of the aforeſaid ſpecies, we ſhall then have $\pi=1$, and conſequently $1-\pi\pi=0$; and ſo, the whole $(\overline{1-\pi\pi.\Pi}\,\text{coſ.}\,\pi z)$ intirely vaniſhing, we have nothing left to compare with the term propoſed (g coſ. z) whereby its effect can be known. Nevertheleſs, it is by means of a term (Π coſ πz) of the ſame ſpecies, entering into the ſeries for the value of $1-w$, that the effect in queſtion muſt be exhibited: for, though indeed the term Π coſ. πz (or Π coſ. z), by receiving the co-efficient $1-\pi\pi$ $(=0)$ intirely vaniſhes, in the firſt line of the general equation, yet, that will not be the caſe in the other parts of the equation; wherein it will be found, affected in the ſame manner with the term B coſ. βz (whereof the determination hath been given, in computing the motion of the *apogee*); which muſt neceſſarily appear to be the caſe, if it be conſidered, that the terms Π coſ. πz and B coſ. βz are alike concerned in the original ſeries $1-\Pi$ coſ. $\pi z-$B coſ. $\beta z-$C coſ. γz &c. from whence the equation itſelf is derived. Hence we not only have a proper term to compare with the given one (g coſ. z), but alſo an eaſy way to diſcover what the reſult will

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* To determine the perturbating forces, ſo as to take in the effect of the ſun's *eccentricity* and *parallax*; let $a=\text{CS}$, $x=\text{CB}$, and $y=\text{CF}$; then BS being $=\sqrt{aa-2ay+xx}$, the force $k\times\frac{\overline{\text{CS}}^2\times\text{CB}}{\overline{\text{BS}}|^3}$ in the direction CB (*vid. p.* 160) will be $=\frac{kx}{a}\times\overline{1-\frac{2y}{a}+\frac{xx}{aa}}\big|^{-\frac{3}{2}}=\frac{kx}{a}\times\overline{1+\frac{3y}{a}}$ &c. and that ($k\times\frac{\text{CS}^3}{\text{SB}^3}-k$) in the direction BH, equal to $k\times\overline{1-\frac{2y}{a}+\frac{xx}{aa}}\big|^{-\frac{3}{2}}-k=k\times\overline{\frac{3y}{a}-\frac{3xx}{2aa}+\frac{15yy}{2aa}}$ &c. which, becauſe $y=x\times$ coſ. BCS, will be reduced to $\frac{kx}{a}\times\overline{1+\frac{3x}{a}\times\text{coſ. BCS}}$ &c. and $k\times\overline{\frac{3x}{a}\times\text{coſ. BCS}+\frac{xx}{4aa}\times\overline{9+15\,\text{coſ.}\ 2\text{BCS}}}$ &c. reſpectively. But the laſt of theſe forces may, again, be reſolved into two others; the one in the direction BC, expreſſed by k coſ. BCS $\times\overline{\frac{3x}{a}\times\text{coſ. BCS}+\frac{xx}{4aa}\times\overline{9+15\,\text{coſ.}\ 2\text{BCS}}}$, $(=\frac{3kx}{2a}\times\overline{1+\text{coſ.}\ 2\text{BCS}}+\frac{3kxx}{8aa}\times\overline{11\,\text{coſ. BCS}+5\,\text{coſ.}\ 3\text{BCS}})$; and the other, in a direction perpendicular thereto, expreſſed by k ſin. BCS $\times\overline{\frac{3x}{a}\times\text{coſ. BCS}+\frac{xx}{4aa}\times\overline{9+15\,\text{coſ.}\ 2\text{BCS}}}$

(=

$(=\frac{3kx}{2a}\times\text{ſin. }2\text{BCS}+\frac{3xx}{8aa}\times\overline{\text{ſin. BCS}+5\text{ ſin. }3\text{BCS}})$. From the former of which, let the force $\frac{kx}{a}\times\overline{1+\frac{3x}{a}\times\text{coſ. BCS}}$. in the oppoſite direction be ſubtracted, and the remainder, $\frac{kx}{2a}\times\overline{1+3\text{coſ.}2\text{BCS}}+\frac{3kxx}{8aa}\times\overline{3\text{ coſ. BCS}+5\text{ coſ. }3\text{BCS}}$, will be the force whereby the gravity of the moon to the earth, is diminiſhed. But the quantity k (which muſt be exterminated) is, at the mean diſtance (d) of the ſun and earth, found to be equal to $d\times\frac{m^2e^4f}{bb\times\text{CA}}$ (*vid. p.* 162); whence, $a^2:d^2::d\times\frac{m^2e^4f}{bb\times\text{CA}}:\frac{d^3}{aa}\times\frac{m^2e^4f}{bb\times\text{CA}}=$ the general value of k; which being ſubſtituted inſtead thereof, the force in the direction of the *radius vector* will become

$$\frac{m^2e^4f}{2bb}\times\frac{d^3}{a^3}\times\frac{\text{CB}}{\text{CA}}\times\overline{1+3\text{coſ.}2\text{BCS}+\frac{\text{CB}}{\text{CS}}\times\overline{\tfrac{9}{4}\text{ coſ. BCS}+\frac{15}{4}\text{ coſ.}3\text{BCS}}};$$

and that in the *perpendicular direction*,

$$=\frac{m^2e^4f}{2bb}\times\frac{d^3}{a^3}\times\frac{\text{CB}}{\text{CA}}\times\overline{3\text{ ſin. }2\text{BCS}+\frac{\text{CB}}{\text{CS}}\times\overline{\tfrac{3}{4}\text{ ſin. BCS}+\frac{15}{4}\text{ ſin. }3\text{ BCS}}}:$$

Where $m^2=0{,}005595$, and $bb=1{,}0097$, nearly (*vid. p.* 174); and where no regard need be had to the value of e^4f, as this quantity, when ſubſtitution is made in the general *formula* (*on p.* 157.) intirely vaniſhes.

be; for ſeeing that $\Pi\ \text{coſ.}\ \pi z$ and $\text{B coſ.}\ \beta z$ have *there* the ſame coefficients, and that the ſum of thoſe belonging to $\text{B coſ.}\ \beta z$ has been already found, in computing the motion of the *apogee*, to be $= -\overline{1-\beta\beta}$, it follows, that $-\overline{1-\beta\beta}.\Pi\ \text{coſ.}\ \pi z$ (or $-\overline{1-\beta\beta}.\Pi\ \text{coſ.}\ z$) will expreſs the value of all the terms of this ſpecies, ariſing in the equation from the introduction of $\Pi\ \text{coſ.}\ z$ into the original, or aſſumed ſeries: whence, by comparing this quantity with the given one, $g\ \text{coſ.}\ z$ (that is, by making $-\overline{1-\beta\beta}.\Pi\ \text{coſ.}\ z + g\ \text{coſ.}\ z = 0$) we get $\Pi = \frac{g}{1-\beta\beta}$; and conſequently $\Pi\ \text{coſ.}\ z = \frac{g}{1-\beta\beta} \times \text{coſ.}\ z$; whereby the required effect of the quantity $g\ \text{coſ.}\ z$, is known.

'Tis true, indeed, that the new term $\Pi\ \text{coſ.}\ z$, thus determined, will introduce an infinity of others; but, of theſe, none will be conſiderable enough to merit the leaſt degree of attention, except (perhaps) ſuch as are expreſſed by the co-ſine of $\overline{1-\beta}.z$; which, becauſe of the ſmallneſs of $1-\beta$, may, for reaſons before ſpecified, produce an effect not to be neglected without a proper examination. To give here the quantity of this effect, it will be neceſſary to obſerve, that, of all the terms in the values of $\overline{1-w}|^{-3}$ and $\overline{1-w}|^{-4}$, *that* which ariſes by the multiplication of $\frac{10\Pi}{e^4} \times \text{coſ.}\ z$ into $2\text{D} \times \text{Coſ.}\ \delta z$, and comes from the part $\frac{10}{e^4} \times \overline{\Pi\ \text{coſ.}\ z + \text{B coſ.}\ \beta z + \text{C coſ.}\ \gamma z + \text{D coſ.}\ \delta z\ \&\text{c.}}|^2$ will ſo much exceed, in its effect, the others wherein Π is concerned, as to render them, in a manner, inconſiderable. But this term $\frac{10\Pi}{e^4}\ \text{coſ.}\ z \times 2\text{D coſ.}\ \delta z$ (or $\frac{10\Pi\text{D}}{e^4}.\ \text{coſ.}\ z \times 2\ \text{coſ.}\ \overline{p-\beta}.z$) is reſolved into $\frac{10\Pi\text{D}}{e^4} \times \text{coſ.}\ \overline{p-\beta+1}.z$, and $\frac{10\Pi\text{D}}{e^4} \times \text{coſ.}\ \overline{p-\beta-1}.z$; whereof the former part only, in which the difference of β and 1 is concerned, is for our purpoſe: from whence (as appears by the obſervations at *p.* 156) a new term expreſſed by $\frac{e\text{P}}{e^4 f} \times \frac{10\Pi\text{D}}{\beta-1} \times \frac{\text{coſ.}\ \overline{\beta-1}.z}{\overline{1-\overline{\beta-1}}|^2}$ (or $\frac{e\text{P}}{e^4 f} \times \frac{10\Pi\text{D}}{1-\beta} \times \text{coſ.}\ \overline{\beta-1}.z$, nearly) will ariſe in the required value of $1-w$. Of the two terms here found, the former gives a very ſmall equation depending on the moon's diſtance from the ſun's *apogee*; and the latter, an equation ſomething more conſiderable, whoſe *argument* is the diſtance of the moon's *apogee* from *that* of the ſun.

By having ſhewn above, that the effect of ſuch terms, or forces, as are proportional to the *co-ſine of the arch z*, is explicable by means of the co-ſines of that arch, and of its multiples, (no leſs than the effects of the other terms that are proportional to the *co-ſines of the multiples thereof)* a very important point is determined: For, ſince it appears thereby, that no terms enter into the equation of the orbit but what by a regular increaſe and decreaſe, do after a certain time return again to their former values, it is evident from thence, that the mean motion, and the greateſt quantities of the ſeveral equations, undergo no change from *gravity*.

THE END.

12 DE 54

ERRATA.

Page 5 line 18, for R read R′; l. 21, for R′ r. R; p. 30. l. 9, for *ba+bb* r. *ba+b*; l. 13, for t r. h; p. 39. l. 14, put the *comma* before ♈O♎; p. 58. l. 11, for PSO r. PSC; p. 59. l. 15, for $\frac{c^3z}{2a}$ r. $\frac{cz}{a}$; l. 22, for F r. N; p. 114, l. 32. r. powers of; p. 127, l. 9, for AGO r. AOG; p. 128. l. 30, for BL. BR; p. 131. l. 23, for BC r. *n*C; p. 138. l. 15, for E*ef* r. E*e*; p. 143, l. 15 dele *to*; p. 152, l. 2, for ββ, r. γγ; p. 160. l. 27, for $\frac{BC}{SC}$ r. $\frac{BC}{Sb}$.

In fig. 18, at the interſection of the circumference of the circle and the right-line F*m*, place an *e*.

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Fig. 1.

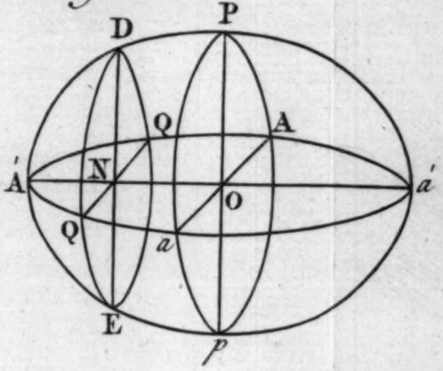


Fig. 2.

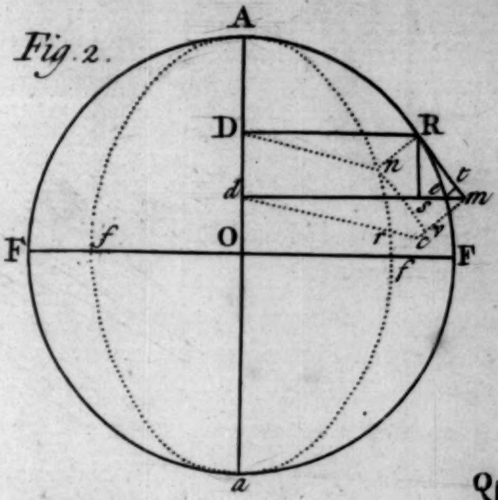
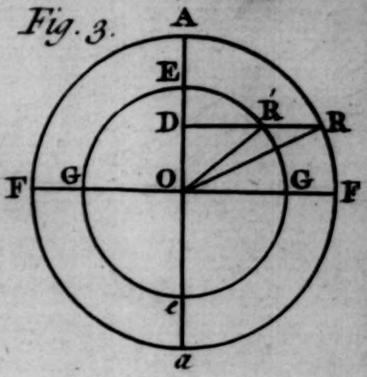


Fig. 3.



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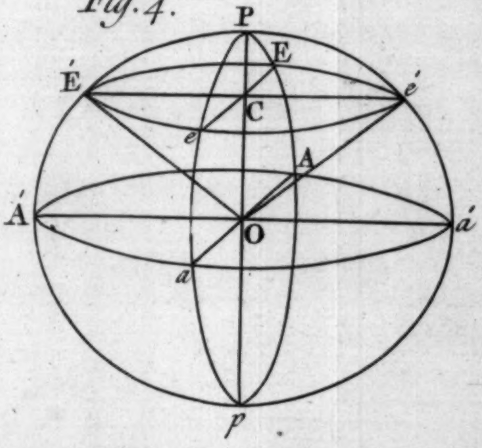
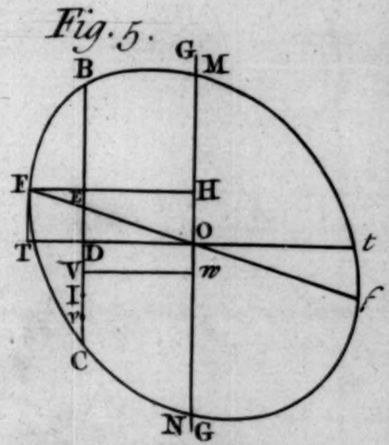


Fig. 5.



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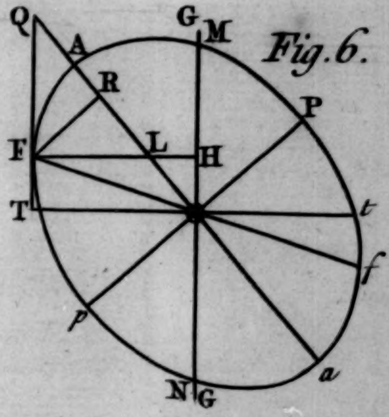

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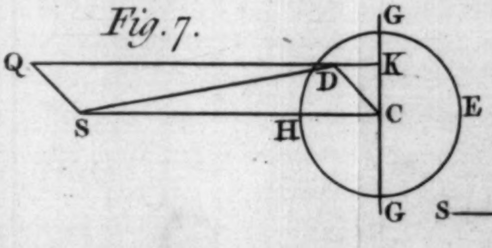

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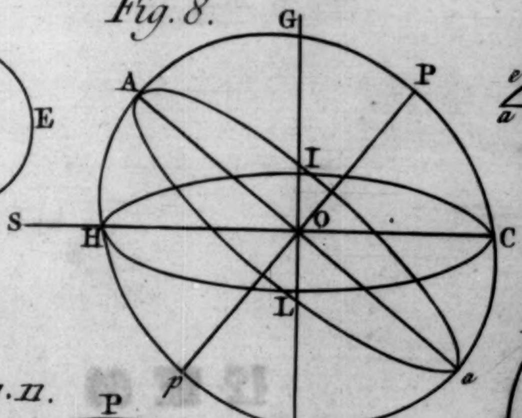


Fig. 9.

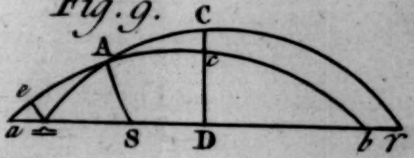


Fig. 10.

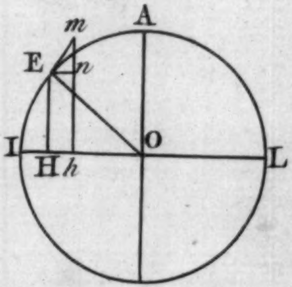


Fig. 11.

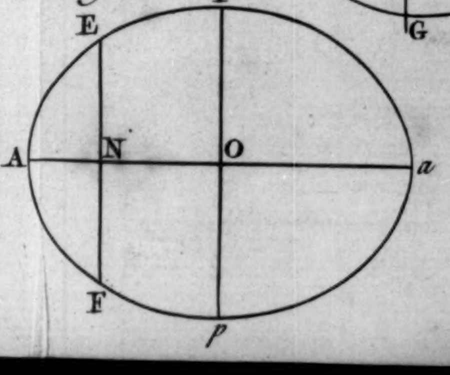
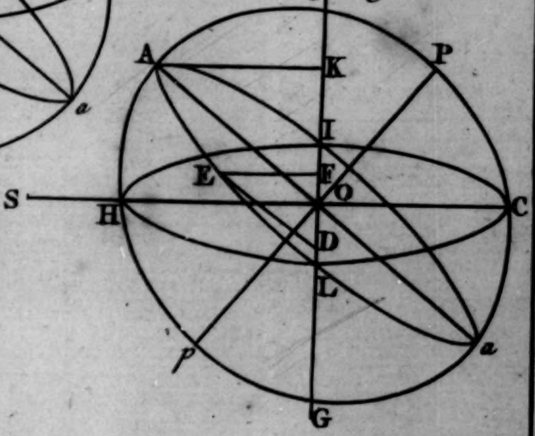


Fig. 12.



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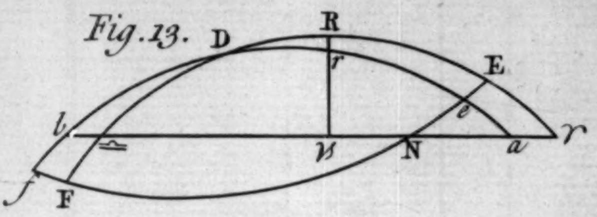
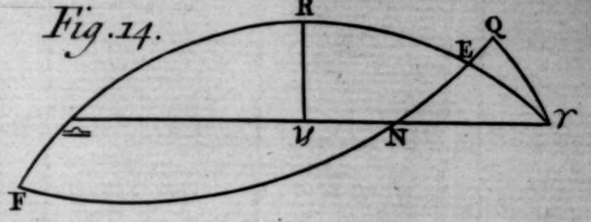


Fig.14.



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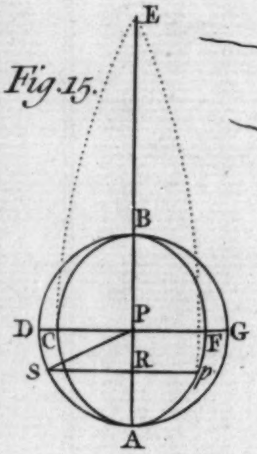


Fig.16.

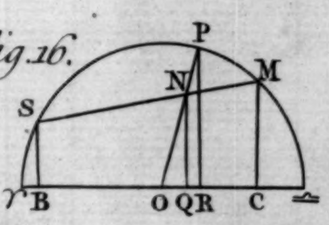


Fig.17.

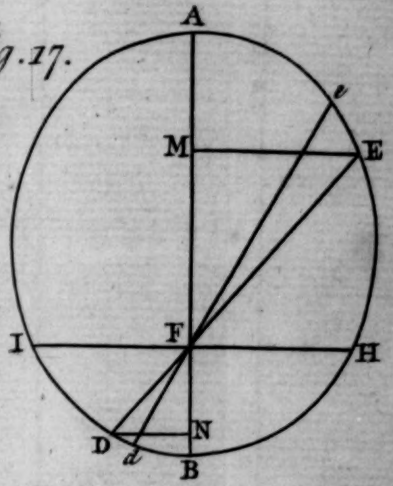


Fig.18.

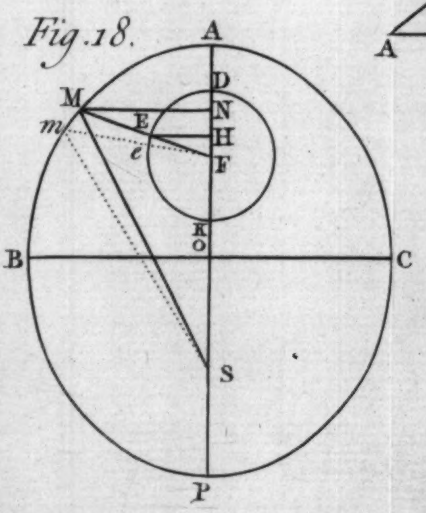
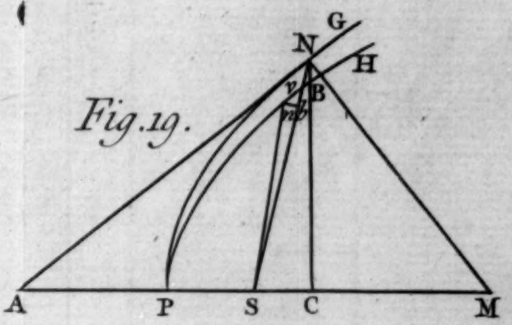


Fig.19.



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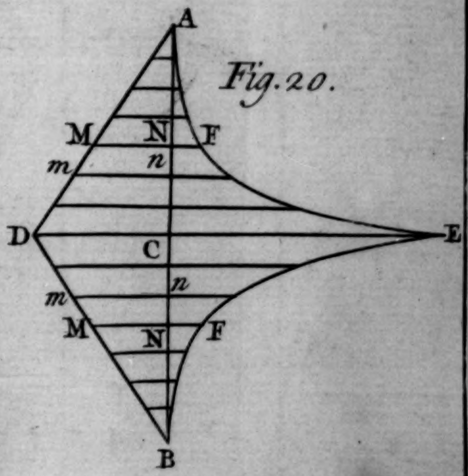

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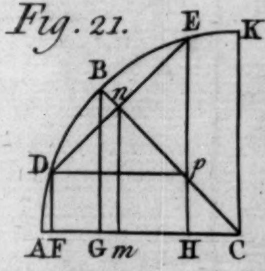

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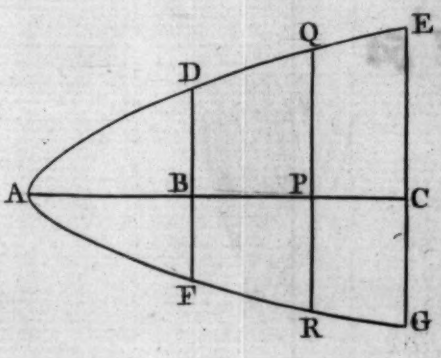

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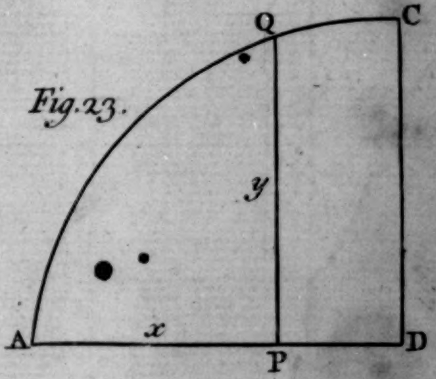

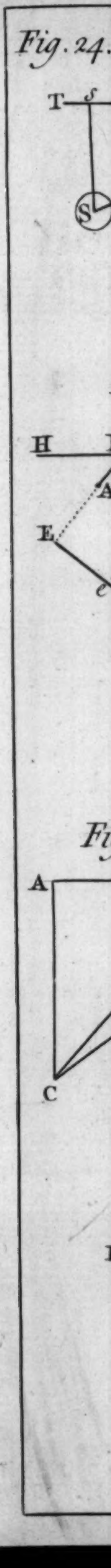

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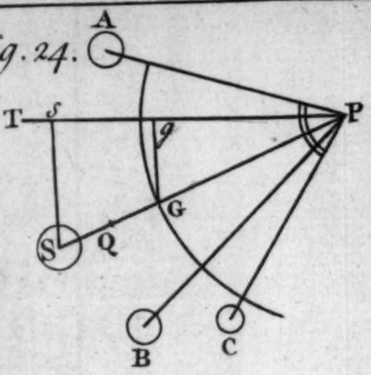
Fig. 24.



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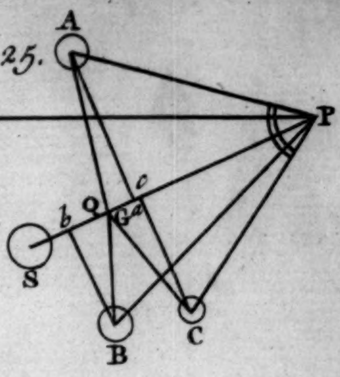

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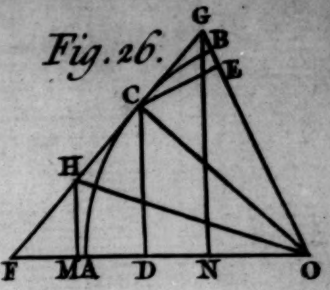


Fig. 27.

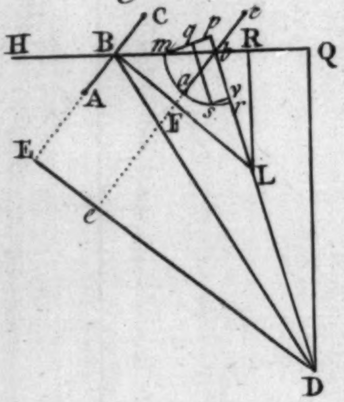
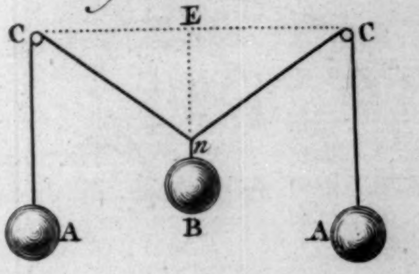


Fig. 28.



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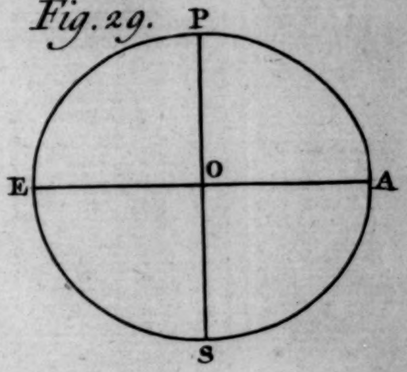

Fig. 30.

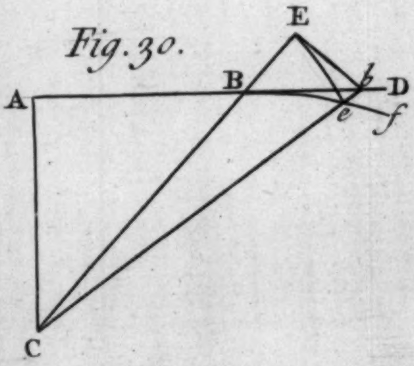

Fig. 31.

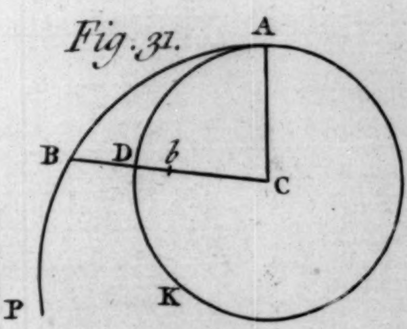

Fig. 32.

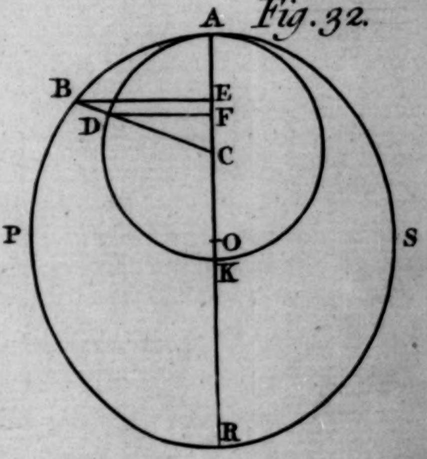

Fig. 33.

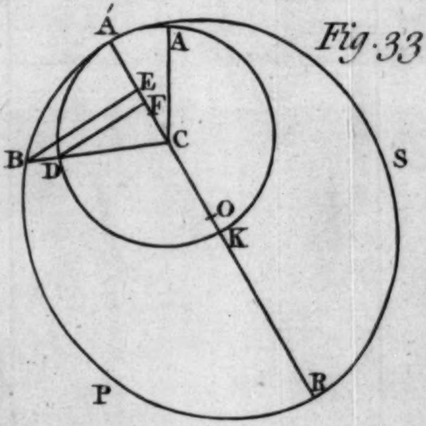

Fig. 34.